Performance Measures and Statistical Quantities of Rake Receivers Using Maximal-Ratio Combining on the IEEE 802.15.3a UWB Channel Model

Kei Hao and John A. Gubner, *Member, IEEE*

***Abstract*— Closed-form expressions for the average signal-to-noise ratio (SNR) and amount of fading are derived for a rake receiver using maximal-ratio combining on the IEEE 802.15.3a ultra-wideband (UWB) channel model. It is also shown that all moments of the SNR and all joint moments of the channel coefficients can be expressed in closed form.**

***Index Terms*— Amount of fading, channel coefficients, Saleh–Valenzuela model, signal-to-noise ratio, ultra-wideband.**

I. Introduction

The ability of ultra-wideband (UWB) systems to resolve a large number of multipaths suggests that rake receivers can be used to exploit diversity. However, combining all paths is prohibitively complex, and evaluating the performance impact of combining only a subset of paths can be a challenging problem, especially under IEEE 802.15.3a UWB channel model.

The IEEE 802.15.3a standards body has developed a modification of the Saleh–Valenzuela multipath channel model as the accepted channel model for UWB investigations. The specification of the Saleh–Valenzuela [11] model and its IEEE 802.15.3a modification [2], [9] are presented in a way that makes them easy to simulate, but difficult to analyze theoretically. However, we have recently shown that if the UWB channel model is treated as a two-dimensional point process, then several statistical quantities of the channel can be computed in closed form [4]. In this paper we study the performance of rake receivers using maximal-ratio combining (MRC) on the IEEE 802.15.3a UWB channel. We derive simple closed-form expressions for the amount of fading (AF) and the average signal-to-noise ratio (SNR) as a function of the number of branches combined. In addition, we show that all moments of the SNR and all joint moments of the channel coefficients can be expressed in closed form, which will be useful to researchers seeking to exploit higher-order statistics, e.g., [15].

The average SNR and the AF are performance measures that describe the behavior of digital communication systems in the presence of fading [13]. The average SNR is the simplest performance measure to compute and requires only the first moment of the SNR. However, this performance measure does not capture all the diversity benefit [13]. The AF was introduced by Charash [3] as a unified measure of the severity of fading and has been generalized by Simon and Alouini [13]. The AF requires higher-order statistics of the combiner output or SNR, and it is shown that it can better capture the diversity benefit than the average SNR [13]. It is also recently shown that the AF depends on the branches combined [1], [14]. The AF has been studied extensively in the literature for several well-known fading channels and combining techniques, e.g., [1], [3], [7], [13], [14]. However, the AF for IEEE 802.15.3a UWB channel model has not been evaluated.

The paper is organized as follows. In Section II, we derive the SNR for rake receivers using MRC for the IEEE 802.15.3a UWB channel model. Section III summarizes our results for the average SNR and the AF. We show that these quantities can be computed in closed form using higher-order statistics of the channel coefficients. Section IV shows that using the joint moments of the channel coefficients, moments of the SNR can be computed in closed form. In particular, the mean and the variance are derived. A numerical example is given in Section V, and the conclusion is in Section VI.

II. Mathematical Model

Frequency-selective fading channels are well modeled by time-varying impulse responses of the form [10]

$$h(t,\tau) = \sum_k G_k(t)\delta(\tau - T_k(t)),$$

where t and τ are the observation time and the application time of the impulse, respectively. Here the $T_k(t)$ are the time-varying path arrival times and the the $G_k(t)$ are the time-varying path gains. Since we focus on indoor environments whose structure changes slowly in comparison with the signaling rate, we use the corresponding time-invariant model,[1]

$$h(\tau) = \sum_k G_k\delta(\tau - T_k).$$

The response of such a channel to a waveform $\xi(t)$ is

$$\hat{\xi}(t) = \sum_k G_k\xi(t - T_k). \tag{1}$$

The authors are with the Department of Electrical and Computer Engineering, University of Wisconsin, Madison, WI 53706–1691 USA (e-mail: khao@wisc.edu, gubner@engr.wisc.edu).

[1] UWB channels exhibit clustering of the arrival times and attenuation of the path gains. To capture these features, IEEE 802.15.3a channel model specifies that paths arrive at times T_k of a kind of cluster process in which the gains G_k are independent marks [2], [9]. For analysis purposes, we have found it convenient to view the pairs (T_k, G_k) as points of a two-dimensional point process [4].

When transmitting a signal $\xi(t)$ of bandwidth W over a channel with delay spread T_{DS}, the response of the tapped-delay-line channel model [10] or virtual-channel model of Sayeed and Aazhang [12] is given by

$$\hat{\xi}(t) \approx \sum_{l=0}^{L} \Phi_l \xi(t - lT_\Delta), \tag{2}$$

where $L \leq L_{\max}$, $L_{\max} \approx T_{DS}W$ is the maximum number of resolvable delays within the delay spread T_{DS}, and $T_\Delta := 1/W$. Note that if $L < L_{\max}$, then the receiver only captures energy from the first $L+1$ resolvable paths or branches and is equivalent to the receiver only observing the waveform on the interval $[0,T)$, where $T := (L+1/2)T_\Delta$. If we let $B_0 := [0, T_\Delta/2)$ and $B_l := [(l-1/2)T_\Delta, (l+1/2)T_\Delta)$ for $l = 1, \ldots, L$, then it can be shown that the virtual channel model (2) is related to (1) via the channel coefficients [6], [12]

$$\Phi_l := \sum_k G_k I_{B_l}(T_k), \tag{3}$$

where $I_{B_l}(\cdot)$ is the indicator function of the set B_l, i.e., $I_{B_l}(t) := 1$ if $t \in B_l$ and $I_{B_l}(t) := 0$ if $t \notin B_l$. Note that under IEEE 802.15.3a UWB channel model, the number of terms in (3) is *random*.

If we also account for additive white Gaussian noise of power spectral density σ_n^2 at the receiver, then any signal detection is based on the waveform $\rho(t) = \hat{\xi}(t) + n(t)$.

In a binary signaling scheme, there are two possible transmitted signals $\xi_i(t)$, $i = 0, 1$, and two received signals $\hat{\xi}_i(t)$. Since the $\xi_i(t)$ are known, conditioned on the channel coefficients Φ_l, the received signals $\hat{\xi}_i(t)$ are also known. This optimum receiver using the tapped-delay-line channel model assuming the channel coefficients can be perfectly estimated is analogous to a rake receiver using maximal-ratio combining [10]. Hence, the conditional probability of error is[2]

$$Q\Big(\sqrt{d^2/4\sigma_n^2}\,\Big), \tag{4}$$

where $Q(x) := \int_x^\infty e^{-t^2/2}/\sqrt{2\pi}\,dt$ is the standard normal complementary cumulative distribution function, and d is the distance between the received waveforms $\hat{\xi}_1$ and $\hat{\xi}_0$. If we put $\hat{\xi}_\Delta(t) := \hat{\xi}_1(t) - \hat{\xi}_0(t)$, then $d^2 = \int |\hat{\xi}_\Delta(t)|^2\,dt$. If we put $\xi_\Delta(t) := \xi_1(t) - \xi_0(t)$, we have

$$d^2 = \sum_{l=0}^{L}\sum_{m=0}^{L} \Phi_l \Phi_m \mathcal{R}((m-l)\Delta),$$

where $\mathcal{R}(t) := \int \xi_\Delta(\theta + t)\xi_\Delta(\theta)\,d\theta$ is the time autocorrelation function of $\xi_\Delta(t)$. If we assume that the duration of the pulses $\xi_\Delta(t)$ is less than T_Δ, then $\mathcal{R}((m-l)T_\Delta) = 0$ for $m \neq l$, and

$$d^2 = \mathcal{R}(0)\sum_{l=0}^{L}\Phi_l^2.$$

Substituting this into (4) and taking the expectation, we obtain the average bit error probability $P_e = \mathsf{E}[Q(\sqrt{\Lambda})]$, where

$$\Lambda := \frac{\mathcal{R}(0)\sum_{l=0}^{L}\Phi_l^2}{4\sigma_n^2}$$

is the output SNR.

[2]In the IEEE 802.15.3a model, the gains G_k are real as is the additive noise.

III. Summary of Results

The SNR Λ is proportional to

$$H_L := \sum_{l=0}^{L}\Phi_l^2.$$

It is shown in [4] that the Φ_l are zero mean and uncorrelated, but are not statistically independent and significantly different from Gaussian distributions [5]. The random variable H_L is the sum of $L+1$ random variables with different distributions. Because the Φ_l are not statistically independent, the distribution of H_L is very difficult to evaluate analytically or compute numerically. Hence, we concentrate on performance measures such as the average output SNR, the AF, and the moments of the SNR at the output of the combiner as a function of number of branches combined.

A. Average SNR

The average SNR as a function of the number of branches combined is given by

$$\text{ASNR}(L) := \mathsf{E}[\Lambda] = \frac{\mathcal{R}(0)\mathsf{E}[H_L]}{4\sigma_n^2}.$$

It is shown in Theorem 1 in Section IV that

$$\begin{aligned}\mathsf{E}[H_L] = \Omega_0\{1 + R\overline{\beta}(T,s_0) + C\overline{\beta}(T,\tau_0) + RC[s_0\overline{\beta}(T,\tau_0) \\ -s_0\overline{\beta}(T, s_0\tau_0/(s_0-\tau_0))e^{-T/s_0}]\},\end{aligned} \tag{5}$$

where $T = (L+1/2)T_\Delta$, and

$$\overline{\beta}(T,\mu) := \int_0^T e^{-t/\mu}\,dt = \mu[1 - e^{-T/\mu}], \tag{6}$$

and the constants R (ray arrival rate), C (cluster arrival rate), τ_0 (cluster decay factor), s_0 (ray decay factor), and Ω_0 are the channel parameters of the IEEE 802.15.3a UWB channel model. The number of branches combined is $L+1$, and T_Δ is defined following (2).

B. Amount of Fading (AF)

The AF as a function of the number of branches combined is defined as [13]

$$\text{AF}(L) := \frac{\mathsf{var}(\Lambda)}{(\mathsf{E}[\Lambda])^2} = \frac{\mathsf{var}(H_L)}{(\mathsf{E}[H_L])^2}. \tag{7}$$

Since H_L is the sum of squared channel coefficients, its variance involves higher-order moments of the channel coefficients. Section IV shows that the variance can be computed using fourth moments and the correlation of the second moments of the channel coefficients. Later it is shown in Sections C and D of the Appendix that the moments and the correlations are in closed form. Hence, the AF as a function of the number of branches combined can be expressed in closed form.

IV. Moments of the SNR

The nth moment of the SNR is given by

$$\mathsf{E}[\Lambda^n] = \frac{\mathcal{R}(0)^n \mathsf{E}[H_L^n]}{(4\sigma_n^2)^n}.$$

It is shown in Section IV-C that the nth moment of H_L involves the joint moments of the channel coefficients. For this reason, first it is necessary to derive the joint characteristic function of the channel coefficients in Section B of the Appendix. Then by computing derivatives of the joint characteristic function, the joint moments of the channel coefficients are shown to be in closed form in Section C of the Appendix. Thus, moments of the SNR can be expressed in closed form.

A. The Mean of H_L

The mean of H_L is given by

$$\mathsf{E}[H_L] = \sum_{l=0}^{L} \mathsf{E}[\Phi_l^2].$$

It is shown in [4] that

$$\begin{aligned}\mathsf{E}[\Phi_l^2] = \Omega_0\{ & I_{B_l}(0) + R\beta(B_l, s_0) + C\beta(B_l, \tau_0) \\ & + RC[\beta(B_l, s_0)\overline{\beta}(a_l, s_0\tau_0/(s_0-\tau_0)) + s_0\beta(B_l, \tau_0) \\ & - s_0\beta(B_l, s_0\tau_0/(s_0-\tau_0))e^{-b/s_0}]\}, \end{aligned} \quad (8)$$

where a_l is the left-hand end point of the interval B_l, $\overline{\beta}(\cdot,\cdot)$ is defined in (6), and for any set B and $\mu > 0$,

$$\beta(B, \mu) := \int_B e^{-t/\mu}\, dt. \quad (9)$$

If B is an interval $[a, b)$, then $\beta(B, \mu) = \mu[e^{-a/\mu} - e^{-b/\mu}]$. Also, $\beta([0,T), \mu) = \overline{\beta}(T, \mu)$. Since all our B_l defined following (2) are intervals, we have (8) in closed form.

Theorem 1: The expectation $\mathsf{E}[H_L]$ is given by (5).

Proof: Suppose we divide the interval $[0, T)$ into the nonoverlapping intervals as done following (2). Each channel coefficient, Φ_l, is defined as the sum of the path gains in the corresponding interval B_l. Since it is shown in [4] that the Φ_l are zero mean and uncorrelated, $\mathsf{E}[\Phi_l^2] = \mathsf{var}(\Phi_l)$, and

$$\begin{aligned}\sum_{l=0}^{L} \mathsf{E}[\Phi_l^2] &= \sum_{l=0}^{L} \mathsf{var}(\Phi_l) = \mathsf{var}\Big(\sum_{l=0}^{L} \Phi_l\Big) \\ &= \mathsf{var}\Big(\sum_k G_k I_{[0,T)}(T_k)\Big), \quad \text{by (3).}\end{aligned}$$

This implies that $\mathsf{E}[H_L]$ can be computed using the right hand side of (8) by replacing B_l with $[0, T)$ and a_l with 0. □

It is also shown in [4] that

$$\lim_{T\to\infty} \mathsf{var}\Big(\sum_k G_k I_{[0,T)}(T_k)\Big) = \Omega_0(1 + Rs_0)(1 + C\tau_0),$$

which is the maximum value of $\mathsf{E}[H_L]$.

B. The Variance of H_L

Using the formula $\mathsf{var}(H_L) = \mathsf{E}[H_L^2] - (\mathsf{E}[H_L])^2$, we have

$$\mathsf{var}(H_L) = \sum_{i=0}^{L} \mathsf{E}[\Phi_i^4] + \sum_{i=0}^{L} \sum_{j=0, j\neq i}^{L} \mathsf{E}[\Phi_i^2 \Phi_j^2] - (\mathsf{E}[H_L])^2. \quad (10)$$

For the IEEE 802.15.3a channel model, it is possible in principle to express all joint moments $\mathsf{E}[\Phi_0^{n_0}\Phi_1^{n_1}\cdots\Phi_L^{n_L}]$ in closed form; this is shown in Section C of the Appendix. For the moments appearing in (10), explicit closed-form expressions are given in Section D of the Appendix. Hence, $\mathsf{var}(H_L)$ can be computed in closed form.

C. Higher-Order Moments of H_L

By the multinomial theorem,

$$\begin{aligned}\mathsf{E}[H_L^n] &= \mathsf{E}\left[\left(\sum_{l=0}^{L} \Phi_l^2\right)^n\right] \\ &= \sum \frac{n!}{n_0! n_1! \cdots n_L!} \mathsf{E}\left[\Phi_0^{2n_0}\Phi_1^{2n_1}\cdots\Phi_L^{2n_L}\right],\end{aligned}$$

where the sum is taken over all multi-indices $n_1, \ldots, n_k$ such that $n_1 + n_2 + \cdots + n_k = n$. As just mentioned above, the joint moments in this last expectation can be computed in closed form, although it would be tedious without a computer for $n > 2$.

V. Numerical Results/Example

As a check of our work, we compare the mean and variance using the closed-form expressions (5) and (10) with Monte–Carlo estimates by simulating 30,000 realizations of the IEEE 802.15.3a UWB channel model. For the numerical example, we consider the channel model CM3 in [2, Table II], namely

$$\Omega_0 = 1/[(1 + Rs_0)(1 + C\tau_0)], \quad C = 0.0667, \quad R = 2.1,$$
$$\tau_0 = 14, \quad s_0 = 7.9, \quad \sigma^2 = \sigma_1^2 + \sigma_2^2, \quad \sigma_1 = \sigma_2 = 3.3941.$$

All times are in nanoseconds.

The examples use the time resolution $T_\Delta = 1$ ns. Fig. 1 shows that the mean (top) and variance (bottom) computed using closed-form expressions and estimates by simulation are very close. Since the mean is normalized to 1, the mean can be used to represent the percentage of expected energy captured by the rake receiver using maximal-ratio combining as a function of L. Note that the average SNR is the same curve as a function of L multiplied by a scale factor.

Fig. 2 shows the AF as a function of the number of branches combined computed using (7). In this channel model, the combiner efficiently mitigates the fading for the first branches combined and the AF remains almost constant beyond 40 branches combined.

VI. Conclusions

We have derived closed-form expressions for the average SNR and AF as a function of the number of branches combined by rake receivers using MRC on the IEEE 802.15.3a UWB channel model. The AF can be used to evaluate how effectively rake receivers using MRC mitigate fading under

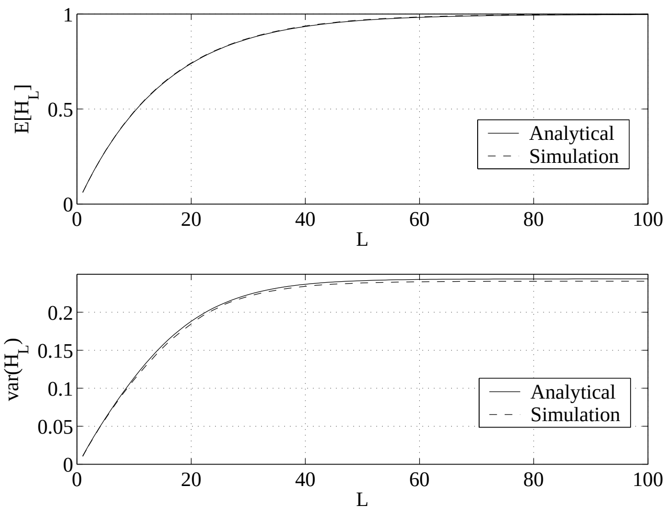

Fig. 1. $\mathsf{E}[H_L]$ (top) and $\mathsf{var}(H_L)$ (bottom) as a function of branches combined.

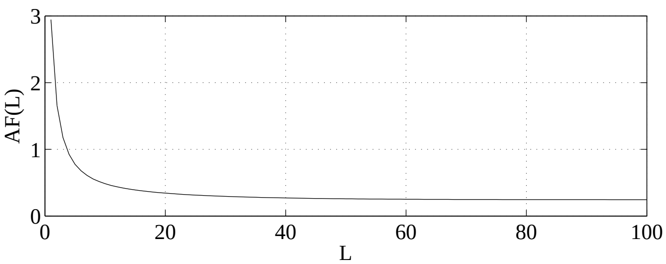

Fig. 2. Amount of fading as a function of the number of branches combined.

the IEEE 802.15.3a UWB channel model. These closed-form expressions quantify the effects of the number of branches combined on these performance measures, and ultimately they will be of great benefit to researchers in the analysis and design of UWB systems. Since the AF requires higher-order statistics of the SNR, it was necessary to derive the joint characteristic function of the channel coefficients. By computing derivatives of the characteristic function, we have also derived closed-form expressions for joint moments of the channel coefficients. Furthermore, we have shown that moments of the SNR can be computed in closed form. Our results should be useful to researchers seeking to exploit higher-order statistics of the channel coefficients as well as the SNR.

APPENDIX

A. Definitions and Notation

In [4] it is shown that the pairs (T_k, G_k) can be regarded as points of a two-dimensional point process. The key idea is the use of the **counting integral**

$$\int_0^\infty \int_{-\infty}^\infty \varphi(s,g)\, N(ds \times dg) := \sum_k \varphi(T_k, G_k), \tag{11}$$

where $N(\cdot)$ is the **counting measure** on $[0,\infty) \times (-\infty,\infty)$ that puts a unit mass at each point (T_k, G_k), where the arrival times T_k and the path gains G_k are specified by the IEEE 802.15.3a model. Note that if we take $\varphi(s,g) = \varphi_l(s,g)$ with $\varphi_l(s,g) = gI_{B_l}(s)$, then Φ_l in (3) is a special case of the (11).

Let $f_{\tau,s}(\cdot)$ denote the density of a path gain arriving at time s that is part of a cluster that started at time τ. Following Saleh and Valenzuela [11, eq. (26)] and Batra *et al.* [2, p. 2126], we assume $f_{\tau,s}(\cdot)$ has second moment[3]

$$\Omega_0 e^{-\tau/\tau_0} e^{-(s-\tau)/s_0}, \tag{12}$$

where τ_0 and s_0 are power-delay time constants and Ω_0 is a scale factor. For the IEEE 802.15.3a model in [2], a $\{\pm 1\}$-valued-Bernoulli$(1/2)$ mixture of lognormal densities is used. This implies that if G has density $f_{\tau,s}(\cdot)$, then $20\log_{10}|G|$ is normal with mean

$$\mu_{\tau,s} := \frac{10}{\ln 10}\Big[\ln\Omega_0 - \tau/\tau_0 - (s-\tau)/s_0 - \Big(\frac{\ln 10}{10}\Big)^2 \frac{\sigma^2}{2}\Big] \tag{13}$$

and variance σ^2. The lognormal mixture density and the lognormal density are related by

$$f_{\tau,s}(x) = \tfrac{1}{2}[f_{|G|,\tau,s}(x) + f_{|G|,\tau,s}(-x)], \tag{14}$$

where $f_{|G|,\tau,s}(\cdot)$ is the lognormal density with parameters $\mu_{\tau,s}$ and σ. The characteristic function of the lognormal mixture random variable is

$$\mathcal{L}_{\tau,s}(\nu) := \mathsf{E}_{\tau,s}[e^{j\nu G}] = \int_{-\infty}^{\infty} e^{j\nu g} f_{\tau,s}(g)\, dg.$$

We also need the following notation. Let $B = [a,b)$,

$$\psi(\tau,\nu,[a,b)) := \begin{cases} \int_{\max(a,\tau)}^{b} 1 - \mathcal{L}_{\tau,s}(\nu)\, ds, & \tau \le b, \\ 0, & \tau > b. \end{cases} \tag{15}$$

and

$$J(\nu,[a,b)) := \int_0^a 1 - e^{-R\psi(\tau,\nu,[a,b))}\, d\tau + \int_a^b 1 - \mathcal{L}_{\tau,\tau}(\nu) e^{-R\psi(\tau,\nu,[a,b))}\, d\tau. \tag{16}$$

To simplify the notation, we define $\psi_l(\tau,\nu) := \psi(\tau,\nu,B_l)$ and $J_l(\nu) := J(\nu,B_l)$, where B_l is defined in Section II.

B. The Joint Characteristic Function

From the definition of the channel coefficients, we can write the joint characteristic function as

$$\Psi(\nu_0,\nu_1,\ldots,\nu_L) = \mathsf{E}[e^{j\sum_{l=0}^{L}\nu_l\Phi_l}].$$

It is shown in [4] that a counting integral of the form (11) can be written as the sum of the three statistically independent terms $\varphi(0,G_0) + \Phi_{r0} + \Phi_\otimes$. Therefore, each channel coefficient defined in (3) is composed of either two or three terms, i.e.,

$$\Phi_0 = \varphi_0(0,G_0) + \Phi_{0,r0} + \Phi_{0,\otimes} = G_0 + \Phi_{0,r0} + \Phi_{0,\otimes},$$

where G_0 is a lognormal mixture random variable, and for $l > 0$, $\Phi_l = \Phi_{l,r0} + \Phi_{l,\otimes}$. By independence, the joint characteristic function can be written as three factors,

$$\Psi(\nu_0,\ldots,\nu_L) = \mathsf{E}[e^{j\nu_0 G_0}]\,\mathsf{E}\Big[\prod_{l=0}^{L} e^{j\nu_l\Phi_{l,r0}}\Big]\,\mathsf{E}\Big[\prod_{l=0}^{L} e^{j\nu_l\Phi_{l,\otimes}}\Big].$$

[3] This is in contrast to [2] and [11]. Their ray processes were defined by taking Poisson processes starting at time zero and then translating them by the arrival time of the initial path in the cluster. The two constructions are equivalent provided we adjust the definition of $f_{\tau,s}(\cdot)$. This is done in (12) where we use $s-\tau$; [2] and [11] would use only s.

The first factor is simply $\mathcal{L}_{0,0}(\nu_0)$ and it is shown in [5] that it can be evaluated numerically. The second factor involves a family of $\Phi_{l,r0}$. As shown in [4], each $\Phi_{l,r0}$ is a shot-noise random variable driven by a two-dimensional Poisson process with intensity function (as a function of (s,g))

$$\lambda_r(s,g|0,G_0) = R f_{0,s}(g), \quad s \geq 0, g \in \mathbb{R}.$$

Hence, we have from [8, Ch. 3] that each factor is given by $\mathsf{E}[\exp(j\nu_l \Phi_{l,r0})] = \exp(-R\psi_l(0,\nu_l))$. Since the channel coefficients lie in disjoint sets, they are independent, thus the second factor is simply the product of these factors, i.e., $\exp\left[-R\sum_{l=0}^{L}\psi_l(0,\nu_l)\right]$. The third factor involves the $\Phi_{l,\otimes}$. As noted in [4, Appendix C], $\Phi_{l,\otimes}$ can be written in the form $\Phi_{l,\otimes} = V_l + P_l$, where, analogous to (11),

$$V_l := \int_0^\infty \int_{-\infty}^\infty \varphi_l(\tau,\gamma) N_1(d\tau \times d\gamma)$$

and

$$P_l := \int_0^\infty \int_{-\infty}^\infty \varphi_l(s,g) N_\times(ds \times dg),$$

where N_1 is the Poisson process with intensity function

$$\lambda_1(\tau,\gamma) := C f_{\tau,\tau}(\gamma), \quad \tau \geq 0, \ \gamma \in \mathbb{R}.$$

and conditioned on the realization of N_1, $N_\times$ is a Poisson process with intensity (cf. [4, Appendix B])

$$m_\times(s,g) = \int_0^\infty \int_{-\infty}^\infty \lambda_r(s,g|\tau,\gamma) N_1(d\tau \times d\gamma).$$

The intensity function $\lambda_r(s,g|\tau,\gamma)$ in this case is given by

$$\lambda_r(s,g|\tau,\gamma) := R f_{\tau,s}(g) I_{[\tau,\infty)}(s),$$

which $\lambda_r(s,g|\tau,\gamma)$ depends on τ but not γ. Conditioned on the realization of N_1, the definition of P_l means that P_l has the conditional characteristic function

$$\mathsf{E}[e^{j\nu_l P_l}|N_1(\cdot)] = \exp\left[-R\int_0^\infty \int_{-\infty}^\infty \psi_l(\tau,\nu_l) N_1(d\tau \times d\gamma)\right],$$

where the last step uses the fact that $\varphi_l(s,g) = gI_{B_l}(s)$. Let us denote this last double integral by $\alpha_l(\nu_l)$. Observe that $\alpha_l(\nu_l)$ looks just like V_l above, except with a different integrand. Then

$$\begin{aligned}\mathsf{E}\left[\prod_{l=0}^{L} e^{j\nu_l \Phi_{l,\otimes}}\right] &= \mathsf{E}\left[\exp\left\{\sum_{l=0}^{L} j\nu_l V_l - R\alpha_l(\nu_l)\right\}\right] \\ &= \exp(-C\bar{J}(\nu_0,\ldots,\nu_L)),\end{aligned} \tag{17}$$

where

$$\bar{J}(\nu_0,\ldots,\nu_L) = \int_0^\infty \int_{-\infty}^\infty \left[1 - \sum_{l=0}^{L} k_l(\tau,\gamma,\nu_l)\right] f_{\tau,\tau}(\gamma)\, d\gamma\, d\tau,$$

and $k_l(\tau,\gamma,\nu_l) := j\nu_l\gamma I_{B_l}(\tau) - R\psi_l(\tau,\nu_l)$. The last step in (17) follows from the fact that N_1 is a Poisson process with intensity $\lambda_1(\tau,\gamma)$ [8, Ch.3].

C. The Statistics of the Channel Coefficients

1) Moments of the Channel Coefficients: We show that moments of the channel coefficients can be computed in closed form. We use this result to compute $\mathsf{var}(H_L)$ in Section D.

To derive the moments of the channel coefficients, we use derivatives of the characteristic function of the channel coefficients. The characteristic function of the channel coefficient Φ_l is given by $\Psi_{\Phi_l}(\nu) = \Psi(0,\ldots,\nu,\ldots,0)$, where only argument $l+1$ of the joint characteristic function is not zero. Hence,

$$\Psi_{\Phi_0}(\nu) = \mathcal{L}_{0,0}(\nu) e^{-R\psi_0(0,\nu)} e^{-CJ_0(\nu)}, \tag{18}$$

and

$$\Psi_{\Phi_l}(\nu) = e^{-R\psi_l(0,\nu)} e^{-CJ_l(\nu)}, \quad l > 0. \tag{19}$$

It is shown in [5] that the characteristic functions of the channel coefficients are even functions of ν, and the probability density functions exist and are also even functions. Hence, odd moments of the channel coefficients are zero. The even moments are

$$\mathsf{E}[\Phi_l^k] = j^{-k}\Psi_{\Phi_l}^{(k)}(\nu)|_{\nu=0}, \quad k \ \text{ even},$$

where $\Psi_{\Phi_l}^{(k)}(\nu)$ is the kth derivative of the characteristic function of Φ_l.

To compute the kth derivative of the characteristic function, we apply Leibniz's rule,

$$(fg)^{(k)}(x) = \sum_{l=0}^{k} \binom{k}{l} f^{(k)}(x) g^{(k-l)}(x).$$

Leibniz's rule implies that the derivatives of the product are in closed form if derivatives of $f(x)$ and $g(x)$ are in closed form. Thus, we need to show each factor in (18) and (19) evaluated at $\nu = 0$ is in closed form.

Since moments of the lognormal random variable are in closed form, it is easy to show that moments of the lognormal mixture random variable are also in closed form. Because the density of the lognormal mixture random variable given by (14) is an even function, odd moments are zero and even moments are given by

$$j^{-m}\mathcal{L}_{\tau,s}^{(m)}(0) = \exp\left[\frac{m\mu_{\tau,s}\ln(10)}{20} + \frac{m^2\sigma^2\ln(10)^2}{2(20^2)}\right], \tag{20}$$

for m even, where $\mu_{\tau,s}$ is given by (13).

The factors in (18) and (19) are exponential functions (except $\mathcal{L}_{0,0}(\nu)$). It is easy to see that the kth derivative of these factors involve the functions themselves and derivatives of the exponents, $\psi_l(0,\nu)$ and $J_l(\nu)$. One can show that $e^{-R\psi_l(0,0)} = 1$ and $e^{-CJ_l(0)} = 1$. Thus, the derivatives of these factors evaluated at $\nu = 0$ are essentially derivatives of $\psi_l(0,\nu)$ and $J_l(\nu)$ evaluated at $\nu = 0$.

The function $\psi_l(\tau,\nu)$ given by (15) is an integral of the characteristic function of the lognormal mixture random variable with respect to the variable s. Because $\mu_{\tau,s}$ is an affine function of the variable s, with a bit of work, one can show that for k even

$$\psi_l^{(k)}(\tau,0) = \begin{cases} -j^k K_\Omega(k)\beta(B_l, 2s_0/k) \\ \cdot\exp[-k\tau(s_0-\tau_0)/(2s_0\tau_0)], & \tau < a_l \\ -j^k K_\Omega(k) 2s_0 \exp[-k\tau/(2\tau_0)]/k \\ +j^k K_\Omega(k) 2s_0 \exp[-b_l k/(2s_0)]/k \\ \cdot\exp[-k\tau(s_0-\tau_0)/(2s_0\tau_0)], & a_l < \tau < b_l \\ 0, & \tau > b_l \end{cases} \tag{21}$$

where

$$K_\Omega(k) = \Omega_0^{k/2}\exp[\sigma^2\ln^2(10)(k^2/2-k)/20^2], \tag{22}$$

$\beta(\cdot,\cdot)$ is given by (9) and $B_l := [a_l, b_l)$. It is straightforward to see that $\psi_l^{(k)}(\tau,\nu)|_{\nu=0} = 0$ if k is odd.

The function $J_l(\nu)$ in (16) involves two integrals. The derivatives of the first integral with respect to ν and evaluated at $\nu=0$ are integrals of derivatives of $e^{-R\psi_l(\tau,\nu)}|_{\nu=0}$, which we have already shown are in closed form given by (21). Since the integrand is an exponential function of τ, the first integral is in closed form. The integrand of the second integral is in product form, therefore we can apply Leibniz's rule again. After applying Leibniz's rule, each term evaluated at $\nu=0$ is the product of two exponential functions given by (20) and (21). Since the integrand is an exponential function of τ, the second integral is also in closed form.

At first glance, it seems that many terms need to be computed to evaluate the moments. However, we show that odd derivatives of the factors in (18) and (19) evaluated at $\nu=0$ are zero. From (20), it is clear that odd derivatives of the factor $\mathcal{L}_{0,0}(\nu)|_{\nu=0}$ are zero.

Theorem 2: The odd derivatives of $e^{-R\psi_l(\tau,\nu)}$ and $e^{-CJ_l(\nu)}$ evaluated at $\nu=0$ are zero.

Proof: This can be proved by induction. Let $\chi_\tau(\nu) := e^{-R\psi_l(\tau,\nu)}$. It is easy to show the first derivative of $\chi_\tau(\nu)$ at the origin is zero. Now suppose that n is an odd number and suppose that all odd derivatives from the first derivative to the nth derivative evaluated at $\nu=0$ are zero. Then $n+2$ is an odd number and applying Leibniz's rule, the $(n+2)$th derivative is given by

$$\chi_\tau^{(n+2)}(\nu) = \sum_{i=0}^{n+1}\binom{n+1}{i}\chi_\tau^{(n+1-i)}(\nu)(-R\psi_l^{(i+1)}(\tau,\nu)).$$

If $i+1$ is odd, then $\psi_l^{(i+1)}(\tau,0)$ is zero. If $i+1$ is even, then i is odd and $\psi_l^{(i+1)}(\tau,0)$ is *not* zero. Since i is odd and $n+1$ is even (because n is odd), then $n+1-i$ is odd. Therefore, $\chi_\tau^{(n+1-i)}(0)$ is zero by the induction hypothesis. Hence, the $(n+2)$th derivative of $\chi_\tau(\nu)$ evaluated at $\nu=0$ is a sum of zeros. The same proof applies to $e^{-CJ_l(\nu)}$. □

Thus, all moments of the Φ_l are in closed form.

2) Joint Moment of the Channel Coefficients: Generalizing the ideas from the previous subsection, the joint moment of the channel coefficients is computed using the joint characteristic function of the channel coefficients, i.e.,

$$\mathsf{E}[\Phi_0^{n_0}\Phi_1^{n_1}\cdots\Phi_L^{n_L}] = \left.\frac{\partial^n\Psi(\nu_0,\ldots\nu_L)}{j^n\partial^{n_0}\nu_0\cdots\partial^{n_L}\nu_L}\right|_{\nu_0=0,\ldots,\nu_L=0}, \tag{23}$$

where $n = n_0+n_1+\cdots+n_L$. To compute the derivatives, we apply the generalized Leibniz rule. It says that if $f_1, f_2, \ldots, f_r$ are functions that are n times differentiable, then

$$\frac{d^n}{dt^n}\prod_{i=1}^r f_i(t) = \sum\frac{n!}{n_1!n_2!\cdots n_r!}\prod_{i=1}^r\frac{d^{n_i}}{dt^{n_i}}f_i(t)$$

where the sum is taken over all multi-indices $n_1,\ldots,n_r$ such that $n_1+n_2+\cdots+n_r = n$.

Let $\chi_l(\nu) := e^{-R\psi_l(0,\nu)}$. It can be shown that

$$\Psi(\nu_0,\ldots,\nu_L) = \mathcal{L}_{0,0}(\nu_0)\chi_0(\nu_0)\cdots\chi_L(\nu_L) \cdot\kappa_J(\nu_0,\ldots,\nu_L), \tag{24}$$

where $\kappa_J(\nu_0,\ldots,\nu_L) = \exp(-CJ_c(\nu_0,\ldots,\nu_L))$. The function $J_c(\nu_0,\ldots,\nu_L)$ can be written as

$$J_c(\nu_0,\ldots,\nu_L) = \int_0^\infty\int_{-\infty}^\infty(1-e^{k_c(\nu_0,\ldots,\nu_L,\tau,\gamma)})f_{\tau,\tau}(\gamma)d\gamma d\tau,$$

where

$$k_c(\nu_0,\ldots,\nu_L,\tau,\gamma) = \sum_{i=0}^L j\nu_i\gamma I_{B_i}(\tau) - R\psi_i(\tau,\nu_i).$$

To compute (23), we apply the generalized Leibniz's rule. We have shown that the first $L+1$ factors of (24) evaluated at $\nu_i=0$ for $i=0,\ldots,L$ are in closed form. To show that the derivatives of $\kappa_J(\nu_0,\ldots,\nu_j)$ evaluated at $\nu_i=0$ for $i=0,\ldots,L$ are also in closed form, we use the same technique discussed earlier, and we use Leibniz's rule one more time. Hence, the joint moments of the channel coefficients can be computed in closed form.

As a special case, the correlation of higher moments is defined as $\mathsf{E}[\Phi_i^p\Phi_j^q]$ for any positive integers p and q. It can be computed by setting $n_i=p$, $n_j=q$ and $n_k=0$ for all k with $k\neq i$ and $k\neq j$.

D. The Variance of H_L

We introduce some notation to simplify the closed-form expression, $\beta_l(u) := \beta(B_l,u)$, $\hat{\beta}_l := \underline{\beta}(B_l, s_0\tau_0/(2s_0-\tau_0))$, $\tilde{\beta}_l := \beta(\underline{B}_l, s_0\tau_0/(2(s_0-\tau_0)))$, and $\underline{\beta}_l := \beta(a_l, s_0\tau_0/(2(s_0-\tau_0)))$, where $\beta(\cdot,\cdot)$ and $\underline{\beta}(\cdot,\cdot)$ are given by (6) and (9) respectively.

It is shown in Section C of the Appendix that the moments of Φ_l are in closed form. The fourth moment of Φ_l for $l\neq 0$ is given by,

$$3R^2\Omega_0^2\beta_l^2(s_0) - 6RC\Omega_0^2\beta_l(s_0)\eta_l + 3C^2\Omega_0^2\eta_l^2 + RK_\Omega(4)\beta_l(s_0/2) - C\bar{\eta}_l \tag{25}$$

where $K_\Omega(4)$ is given by (22),

$$\eta_l = -\beta_l(\tau_0) - R[\beta_l(s_0)\bar{\beta}(a_l, s_0\tau_0/(s_0-\tau_0)) - s_0(\beta_l(\tau_0) - e^{-b_l/s_0}\beta_l(s_0\tau_0/(s_0-\tau_0)))],$$

and

$$\bar{\eta}_l = -K_\Omega(4)\beta_l(\tau_0/2) - 6R\Omega_0^2[s_0(\beta_l(\tau_0/2) - e^{-b_l/s_0}\hat{\beta}_l] - 3R^2\Omega_0^2[\beta_l^2(s_0)\bar{\beta}_l + s_0^2(\beta_l(\tau_0/2) - 2e^{-b_l/s_0}\hat{\beta}_l + e^{-2b_l/s_0}\tilde{\beta}_l] - RK_\Omega(4)[\beta_l(s_0/2)\bar{\beta}_l - s_0/2(\beta_l(\tau_0/2) - e^{-2b_l/s_0}\tilde{\beta}_l].$$

To compute the fourth moment of Φ_0, we use the same expression (25) by replacing B_l with B_0 and adding the following two terms, $K_\Omega(4) + 6\Omega_0^2[R\beta_0(s_0) - C\eta_0]$.

The correlation of the second moment (for $i \neq 0$ and $j \neq 0$) can be computed as

$$\mathsf{E}[\Phi_i^2 \Phi_j^2] = \left. \frac{\partial^4 \Psi_{\Phi_i,\Phi_j}(\nu_i,\nu_j)}{j^4 \partial^2 \nu_i \partial^2 \nu_j} \right|_{\nu_i=0,\nu_j=0}$$

where $\Psi_{\Phi_i,\Phi_j}(\nu_i,\nu_j)$ is the joint characteristic function of channel coefficients Φ_i and Φ_j. With a bit of work, it can be shown that $\mathsf{E}[\Phi_i^2 \Phi_j^2]$ is given by

$$\Omega_0^2 [R^2 \beta_i(s_0)\beta_j(s_0) - RC\beta_i(s_0)\eta_j - RC\beta_j(s_0)\eta_i + C^2 \eta_i \eta_j + R\beta_j(s_0)[C\hat{\beta}_i + R\beta_i(s_0)\beta_i + RCs_0(\hat{\beta}_i - R\tilde{\beta}_i e^{-b_i/s_0})]],$$

where $B_i := [a_i, b_i)$ and $B_j = [a_j, b_j)$ with $a_i < b_i \leq a_j < b_j$. In the case $\mathsf{E}[\Phi_0^2 \Phi_j^2]$ for $j \neq 0$, it is the same expression by replacing B_i with B_0, keeping the same B_j, and adding the following terms $\Omega_0^2 [R\beta_j(s_0) - C\eta_j]$.

References

[1] M.-S. Alouini and M. K. Simon, "Performance of generalized selection combining over Weibull fading channels," *IEEE Vehicular Technology Conference*, vol. 3, pp. 1735–1739, Oct. 2001.

[2] A. Batra, J. Balakrishnan, G. R. Aiello, J. R. Foerster, and A. Dabak, "Design of a multiband OFDM system for realistic UWB channel environments," *IEEE Trans. Microwave Theory Tech.*, vol. 52, no. 9, pp. 2123–2138, Sept. 2004.

[3] U. Charash, "Reception through Nakagami fading multipath channels with random delays," *IEEE Trans. Commun.*, vol. 27, pp. 657–670, Apr. 1979.

[4] J. A. Gubner and K. Hao, "Analysis of the IEEE 802.15.3a UWB channel model," submitted to *IEEE J. Select. Areas Commun.*, Mar. 2005. `http://homepages.cae.wisc.edu/~gubner/jsac2col.pdf`

[5] K. Hao and J. A. Gubner, "The distribution of sums of path gains in the IEEE 802.15.3a UWB channel model," submitted to *IEEE Trans. Wireless Commun.*, June 2005. `http://homepages.cae.wisc.edu/~gubner/`

[6] H. Hashemi, "Impulse response modeling of indoor radio propagation channels," *IEEE J. Select. Areas Commun.*, vol. 11, no. 7, pp. 967–978, Sept. 1993.

[7] G. K. Karagiannidis, D. A. Zogas and S. A. Kotsopoulos, "Statistical properties of the EGC output SNR over correlated Nakagami-m fading channels," *IEEE Trans. Wireless Commun.*, vol. 3, pp. 1764–1769, Sept. 2004.

[8] J. F. C. Kingman, *Poisson Processes*. Oxford, U.K.: Clarendon, 1993.

[9] A. F. Molisch, J. R. Foerster, and M. Pendergrass, "Channel models for ultrawideband personal area networks," *IEEE Wireless Communications*, vol. 10, no. 6, pp. 14–21, Dec. 2003.

[10] J. G. Proakis, *Digital Communications*, 4th ed. Boston: McGraw-Hill, 2001.

[11] A. A. M. Saleh and R. Valenzuela, "A statistical model for indoor multipath propagation," *IEEE J. Select. Areas Commun.*, vol. SAC-5, no. 2, pp. 128–137, Feb. 1987.

[12] A. M. Sayeed and B. Aazhang, "Joint multipath-Doppler diversity in mobile wireless communications," *IEEE Trans. Commun.*, vol. 47, pp. 123–132, Jan. 1999.

[13] M. K. Simon and M. Alouini, *Digital Communication over Fading Channels: A Unified Approach to Performance Analysis*, 2nd ed. New York: Wiley, Jan. 2005.

[14] M. Z. Win and J. H. Winters, "Analysis of hybrid selection/maximal-ratio combining in Rayleigh fading," *IEEE Trans. Commun.*, vol. 47, pp. 1773–1776, Dec. 1999.

[15] M. Z. Win, R. K. Mallik and G. Chrisikos, "Higher order statistics of antenna subset diversity," *IEEE Trans. Wireless Commun.*, vol. 2, pp. 871–875, Sept. 2003.