Linear Estimation of Signals Transmitted over the IEEE 802.15.3a UWB Channel

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***Abstract*— When transmitting a known signal over a multipath channel, the received waveform is a stochastic process due to the random nature of the multipath arrival times and gains. Any program aimed at linear estimation of the random waveform requires its correlation function. Explicit formulas are given for this correlation function when the multipath arrival times and gains are described by the recently developed IEEE 802.15.3a ultra-wideband (UWB) channel model. When the transmitted signal is a Gaussian monocycle, the correlation function can be obtained in closed form. More generally, structural properties of the correlation function are derived, and it is shown that linear estimation of UWB quantities is tractable.**

***Index Terms*— Correlation function, Gaussian monocycle, linear minimum mean squared error estimation, signal estimation, ultra-wideband (UWB).**

I. Introduction

If a waveform $\xi(t)$ is transmitted over a multipath channel, the receiver observes the waveform

$$Y(t) = X(t) + W(t),$$

where $W(t)$ is additive white Gaussian noise with zero mean and power spectral density σ_W^2, and

$$X(t) = \sum_i G_i \xi(t - T_i). \tag{1}$$

In this paper, we assume that the random multipath arrival times and gains are specified by the IEEE 802.15.3a UWB channel model as described in [1], [3], [9], [10]. For the purpose of second order analysis, this model is completely characterized by the channel parameters

$$\Omega_0, C, R, \tau_0, \text{ and } s_0, \tag{2}$$

where Ω_0 is a scale factor, C is the cluster arrival rate, R is the ray arrival rate, and τ_0 and s_0 are power-delay time constants.

If the signal $X(t)$ were deterministic, then digital communication over the UWB channel would be reduced to M-ary detection. However, the signal $X(t)$ is random due to the multipath nature of the channel. One approach would be to compute linear minimum mean squared error estimates of $X(t)$ based on $Y(t)$. The key quantity in this or any similar program is the correlation function $\Gamma_X(t,\theta) := \mathsf{E}[X(t)X(\theta)]$ and its associated operator on waveforms b,

$$\big(\widetilde{\Gamma}_X b\big)(t) := \int_{-\infty}^{\infty} \Gamma_X(t,\theta) b(\theta)\, d\theta.$$

Unfortunately, the complicated structure of the IEEE 802.15.3a UWB channel model makes the determination of $\Gamma_X(t,\theta)$ a formidable task. In this paper, we use recent results in [6] to obtain explicit formulas for the correlation function and its associated operator in terms of the channel parameters (2) and the transmitted waveform ξ. A common choice for ξ in UWB studies, e.g., [2], [12], is the Gaussian monocycle, which has the form $(1-t^2)e^{-t^2/2}$. In this case, we show that the correlation function can be obtained in closed form.

Since the front end of a receiver typically correlates the incoming signal with various template waveforms, say $b_1, \ldots, b_K$, the matrices with $i\,j$ elements

$$\langle b_j, b_i \rangle := \int_{-\infty}^{\infty} b_j(t) b_i(t)\, dt \quad \text{and} \quad \big\langle \widetilde{\Gamma}_X b_j, b_i \big\rangle$$

are of central importance. In this paper, we exploit our formulas for the correlation function and its associated operator to derive structural properties of the second matrix. We show that under conditions met in practice, it is multidiagonal and therefore sparse. Furthermore, since few of its nonzero entries are distinct, their evaluation does not require much computation. We also show that in a special case, the matrix is guaranteed to be invertible.

II. Assumptions and Notation

We assume that the template waveforms $b_1, \ldots, b_K$ are linearly independent. Hence, these waveforms are a basis for the subspace

$$\mathscr{B} := \operatorname{span}\{b_1, \ldots, b_K\}.$$

The projection [8, pp. 160–161] of a waveform x onto $\mathscr{B}$ is given by $B(B^*B)^{-1}B^*x$, where B is the linear operator defined by

$$(B\mathbf{c})(t) := \sum_{k=1}^{K} c_k b_k(t), \quad \mathbf{c} := [c_1, \ldots, c_K]' \in \mathbb{R}^K,$$

where the $'$ denotes the transpose, and B^* is the adjoint of B, given by

$$B^*x = \big[\langle x, b_1 \rangle, \ldots, \langle x, b_K \rangle\big]'.$$

It is easy to check that B^*B is the $K \times K$ matrix whose $i\,j$ element is $\langle b_j, b_i \rangle$.

If we correlate $Y(t) = X(t) + W(t)$ with each template $b_k(t)$ and collect the results into a column vector $\mathbf{Y}$, we can write

$$\mathbf{Y} = \mathbf{X} + \mathbf{W},$$

where $\mathbf{Y} := B^*Y$, $\mathbf{X} := B^*X$, and $\mathbf{W} := B^*W$. The correlation matrix of $\mathbf{W}$ is $\Gamma_{\mathbf{W}} := \mathsf{E}[\mathbf{W}\mathbf{W}'] = \sigma_W^2 B^*B$, while

$$\Gamma_{\mathbf{X}} = B^* \widetilde{\Gamma}_X B,$$

$\Gamma_{\mathbf{XY}} = \Gamma_{\mathbf{X}}$, and $\Gamma_{\mathbf{Y}} = \Gamma_{\mathbf{X}} + \Gamma_{\mathbf{W}}$. In particular, note that the $i\,j$ element of $\Gamma_{\mathbf{X}}$ is given by $\big\langle \widetilde{\Gamma}_X b_j, b_i \big\rangle$.

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Remark: The linear minimum mean squared error estimate of $\mathbf{X}$ based on $\mathbf{Y}$ is given by $A\mathbf{Y}$, where A is the $K\times K$ matrix that minimizes $\mathsf{E}[\|\mathbf{X}-A\mathbf{Y}\|^2]$ and $\|\cdot\|$ is the Euclidean norm on $\mathrm{I\!R}^K$. This is a standard textbook problem, e.g., [5], and the optimal A is any solution of the normal equations $A\Gamma_{\mathbf{Y}} = \Gamma_{\mathbf{XY}}$. It is easy to see that $\Gamma_{\mathbf{Y}} = B^*\big(\sigma_W^2 I + \widetilde{\Gamma}_X\big)B$ is invertible since the operator $\sigma_W^2 I + \widetilde{\Gamma}_X$ is positive definite and since B is nonsingular on account of the linear independence of the templates b_k. We can therefore write $A = \Gamma_{\mathbf{X}}\Gamma_{\mathbf{Y}}^{-1}$.

III. The Correlation Function $\Gamma_X(t,\theta)$

A straightforward generalization of [6, eq. (21)] shows that

$$\Gamma_X(t,\theta) = \Omega_0[\xi(t)\xi(\theta) + Q_\xi(t,\theta)], \tag{3}$$

where[1]

$$Q_\xi(t,\theta) := \int_0^\infty \xi(t-s)\xi(\theta-s)q(s)\,ds, \tag{4}$$

and

$$q(s) := Re^{-s/s_0} + Ce^{-s/\tau_0} + CR\frac{s_0\tau_0}{s_0-\tau_0}\big[e^{-s/s_0} - e^{-s/\tau_0}\big].$$

Note that the last term is nonnegative whether $s_0 < \tau_0$ or $s_0 \geq \tau_0$, and thus $q(s) > 0$ for all s. Furthermore,

$$q(t-\theta) = e^{-t/s_0}q_{s_0}(\theta) + e^{-t/\tau_0}q_{\tau_0}(\theta)$$

where

$$q_{s_0}(\theta) = Re^{\theta/s_0}\left[1 + C\frac{s_0\tau_0}{s_0-\tau_0}\right], \tag{5}$$

and

$$q_{\tau_0}(\theta) = Ce^{\theta/\tau_0}\left[1 - R\frac{s_0\tau_0}{s_0-\tau_0}\right]. \tag{6}$$

We now mention several cases in which $Q_\xi(t,\theta)$, and hence the correlation function $\Gamma_X(t,\theta)$, can be obtained in closed form.

Observe that since $q(s)$ is a sum of decaying exponentials, it suffices to consider integrals of the form

$$\int_0^\infty \xi(t-s)\xi(\theta-s)e^{-s/\mu}\,ds, \tag{7}$$

where $\mu = s_0$ or $\mu = \tau_0$. Hence, if ξ is piecewise constant, $Q_\xi(t,\theta)$ can be obtained in closed form. More generally, if ξ is a polynomial, or even piecewise polynomial, repeated integration by parts allows (7) to be computed in closed form. More interestingly, a straightforward but tedious calculation shows that when ξ is the Gaussian monocycle, $\xi(t) = (1-t^2)\exp(-t^2/2)$, (7) is given by

$$Q_\xi(t,\theta) = e^{-(t^2+\theta^2)}e^{\alpha^2/2}[F_\alpha(0)d_0(t,\theta) + F_\alpha(1)d_1(t,\theta)],$$

where

$$F_\alpha(0) = Q(-\alpha/\sqrt{2})\sqrt{4\pi}, \quad F_\alpha(1) = e^{-\alpha^2/2}/2,$$

$Q(x) = (\sqrt{2\pi})^{-1}\int_x^\infty e^{-t^2/2}\,dt$ is the standard Gaussian complementary cumulative distribution function, $\alpha = t+\theta-1/\mu$, and $d_0(t,\theta)$ and $d_1(t,\theta)$ are combinations of α and $a_i(t,\theta)$ for $i = 0,\ldots,4$ which are the coefficients of the polynomial $(1-(t-s)^2)(1-(\theta-s)^2)$ as a function of s. More generally, it can be shown that if $\xi(t) = p(t)e^{-t^2/2}$, where $p(t)$ is a polynomial, then (7) can be obtained in closed-form using appropriately defined coefficients $F_\alpha(k)$ and $d_k(t,\theta)$.

[1] The lower limit in the integral defining $Q_\xi(t,\theta)$ stems from the fact that under the IEEE 802.15.3a model, the path arrival times T_i in (1) are nonnegative.

IV. Properties and Structure of $\Gamma_{\mathbf{X}}$

Using (3), we see that

$$(\Gamma_{\mathbf{X}})_{ij} = \big\langle\widetilde{\Gamma}_X b_j, b_i\big\rangle = \Omega_0\big[\langle b_j,\xi\rangle\langle\xi,b_i\rangle + \big\langle\widetilde{Q}_\xi b_j, b_i\big\rangle\big], \tag{8}$$

where

$$\big(\widetilde{Q}_\xi b\big)(t) := \int_{-\infty}^\infty Q_\xi(t,\theta)b(\theta)\,d\theta.$$

The structure of $X(t)$ in (1) suggests that we take $b_k(t) = \beta(t-t_k)$ for some waveform β and some fixed shifts $0 \leq t_1 < \cdots < t_K < \infty$ (we continue to assume that the b_k are linearly independent). For example, we could take $\beta(t) = \xi(t)$ and use uniformly spaced shifts [7], [11], but we do not do so yet. When $b_k(t) = \beta(t-t_k)$, it is convenient to introduce the temporal correlation function

$$\Gamma_\beta(\tau) := \int_{-\infty}^\infty \beta(t+\tau)\beta(t)\,dt$$

and the temporal cross-correlation function

$$\Gamma_{\xi\beta}(\tau) := \int_{-\infty}^\infty \xi(t+\tau)\beta(t)\,dt.$$

Then the elements of $\Gamma_{\mathbf{W}} = \sigma_W^2 B^*B$ are just $\sigma_W^2\langle b_j,b_i\rangle = \sigma_W^2\Gamma_\beta(t_i-t_j)$. Similarly, in (8), $\langle\xi,b_i\rangle = \Gamma_{\xi\beta}(t_i)$, and

$$\big\langle\widetilde{Q}_\xi b_j, b_i\big\rangle = \int_0^\infty q(s)\Gamma_{\xi\beta}(t_i-s)\Gamma_{\xi\beta}(t_j-s)\,ds. \tag{9}$$

If we consider the Gaussian monocycle for both $\xi(t)$ and $\beta(t)$, then $\Gamma_{\xi\beta}(\tau) = \Gamma_\beta(\tau) = (1-\tau^2+\tau^4/12)\exp(-\tau^2/4)$, which is essentially the product of a polynomial and Gaussian density. Hence, in this case, (9) has the same structure as (4), and as noted in the discussion following (7), can be expressed in closed form.

A. $\Gamma_{\mathbf{X}}$ Is Invertible If $\beta = \xi$

On account of (8), observe that $\Gamma_{\mathbf{X}}$ is positive definite if for $b \in \mathscr{B}$, $\big\langle\widetilde{Q}_\xi b, b\big\rangle = 0$ implies $b = 0$. Since

$$\big\langle\widetilde{Q}_\xi b, b\big\rangle = \int_0^\infty q(s)\left|\int_{-\infty}^\infty \xi(t-s)b(t)\,dt\right|^2 ds,$$

$\big\langle\widetilde{Q}_\xi b, b\big\rangle = 0$ implies that the inner integral must be zero for almost every $s \geq 0$. For reasonable waveforms $\xi(t)$ and $b(t)$, e.g., having finite energy [4, p. 232, Prop. 8.8], the inner integral is a continuous function of s, in which case it must be zero for all $s \geq 0$. Let

$$\Xi := \operatorname{span}\{\xi(\cdot - s), s \geq 0\}.$$

The condition $\langle\xi(\cdot - s), b\rangle = 0$ for all $s \geq 0$ says that $b \in \Xi^\perp$, the orthogonal complement of Ξ. If we take $\mathscr{B} \subset \Xi$, then $\Xi^\perp \subset \mathscr{B}^\perp$. In this case, $b \in \Xi^\perp$ implies $b \in \mathscr{B}^\perp$. Since $b \in \mathscr{B}$, this means that b is orthogonal to itself; i.e., $\langle b,b\rangle = 0$, which implies $b = 0$. If we take $b_k(t) = \beta(t-t_k)$ and $\beta(t) = \xi(t)$, then $\mathscr{B} \subset \Xi$ and $\Gamma_{\mathbf{X}}$ will be invertible.

B. $\Gamma_{\mathbf{X}}$ Is Sparse Multidiagonal If β and ξ Are Finite-Duration Signals

We continue to assume $b_k(t) = \beta(t - t_k)$, but we do *not* assume $\beta(t) = \xi(t)$.

Using a change of variable in (9) yields

$$\left\langle \widetilde{\Gamma}_X b_j, b_i \right\rangle = \int_{-\infty}^{t_j} q(t_j - \theta)\Gamma_{\xi\beta}(t_i - t_j + \theta)\Gamma_{\xi\beta}(\theta)\, d\theta.$$

Applying (5) and (6) shows that $\left\langle \widetilde{\Gamma}_X b_j, b_i \right\rangle$ is equal to

$$\begin{aligned} &e^{-t_j/s_0}\int_{-\infty}^{t_j} q_{s_0}(\theta)\Gamma_{\xi\beta}(t_i - t_j + \theta)\Gamma_{\xi\beta}(\theta)\, d\theta \\ &\quad + e^{-t_j/\tau_0}\int_{-\infty}^{t_j} q_{\tau_0}(\theta)\Gamma_{\xi\beta}(t_i - t_j + \theta)\Gamma_{\xi\beta}(\theta)\, d\theta. \end{aligned} \tag{10}$$

Now suppose that ξ and β both live on $[0,\Delta]$. Then $\Gamma_{\xi\beta}(\tau) = 0$ for $|\tau| \geq \Delta$. This implies that the lower limits in (10) can be changed to $-\Delta$, and the upper limits can be changed to $\min(t_j,\Delta)$; i.e., (10) becomes

$$\begin{aligned} &e^{-t_j/s_0}\int_{-\Delta}^{\min(t_j,\Delta)} q_{s_0}(\theta)\Gamma_{\xi\beta}(t_i - t_j + \theta)\Gamma_{\xi\beta}(\theta)\, d\theta \\ &\quad + e^{-t_j/\tau_0}\int_{-\Delta}^{\min(t_j,\Delta)} q_{\tau_0}(\theta)\Gamma_{\xi\beta}(t_i - t_j + \theta)\Gamma_{\xi\beta}(\theta)\, d\theta. \end{aligned} \tag{11}$$

It then follows that (11) is zero for $|t_i - t_j| \geq 2\Delta$. This implies that $\Gamma_{\mathbf{X}}$ is a multidiagonal matrix. To be more precise about this, let

$$\delta := \min_{2\leq k\leq K}(t_k - t_{k-1}),$$

and for convenience, assume that

$$\Delta = n\delta$$

for some positive integer n. Then for $i > j$,

$$t_i - t_j = \sum_{k=j+1}^{i} t_k - t_{k-1} \geq (i-j)\delta,$$

and we see that $|t_i - t_j| < 2\Delta$ implies $|i - j| < 2\Delta/\delta = 2n$. Hence, $\Gamma_{\mathbf{X}}$ is nonzero only on the main diagonal and on the $2n-1$ upper and lower diagonals. A similar analysis of $\langle b_j, b_i\rangle = \Gamma_\beta(t_i - t_j)$ shows that B^*B is multidiagonal with nonzero entries only on the main diagonal and on the $n-1$ upper and lower diagonals.

Without loss of generality, we restrict attention to $t_j \leq t_i$ and point out some simplifications in (11). Write

$$t_j = \sum_{k=2}^{j} t_k - t_{k-1} \geq (j-1)\delta.$$

Then the largest value of j with $t_j < \Delta$ satisfies $j < 1 + \Delta/\delta = 1 + n$. For each such j, we must compute (11) for $i = j,\ldots,n$. There are at most $n(n+1)/2$ such times that (11) must be computed. (If $t_1 \geq \Delta$, there are no t_j with $t_j < \Delta$.) For $t_j \geq \Delta$, the integrals in (11) depend on t_i and t_j only through their difference. For example, if $t_k - t_{k-1} = \delta$ for all k (uniformly spaced basis functions), then we only have to compute the integrals in (11) for $t_i - t_j = (i-j)\delta$ for $0 \leq i - j < 2n$. Hence, for uniformly spaced basis functions, there are at most $n(n+1)/2 + 2n$ times that we need to evaluate (11).

Example: Consider a UWB system in which a signal ξ of duration $\Delta = 1$ ns is used. Let $\delta = \Delta/2$ so that $n = 2$. In a line-of-sight channel such as CM1 [3], we might want the basis functions to cover the first 40 ns, which corresponds to $K = 80$. Thus, although $\Gamma_{\mathbf{X}}$ has 6400 entries, only the main diagonal and the 3 upper and lower diagonals are nonzero (fewer than $7K = 560$ or 9% nonzero values). More importantly, (11) has to be evaluated at most 7 times.

V. Conclusion

We have considered received UWB signals $X(t)$ in (1) when the multipath arrival times and gains are given by the IEEE 802.15.3a channel model. We have presented explicit formulas for the correlation function $\Gamma_X(t,\theta) = \mathsf{E}[X(t)X(\theta)]$ in terms of the transmitted waveform ξ and the channel parameters (2). We showed that in many cases of interest, including the Gaussian monocycle, $\Gamma_X(t,\theta)$ can be found in closed form. Even when the correlation function is not available in closed form, we showed that under fairly general conditions, the matrix $\Gamma_{\mathbf{X}}$ is multidiagonal with few distinct nonzero entries to be computed. We also showed that if the receiver correlates its input with shifts of the transmitted pulse ξ, then $\Gamma_{\mathbf{X}}$ must be invertible.

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