Upper Bounds for the Average Error Probability of a Time-Hopping Wideband System

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***Abstract*— Ultra-wideband technology has been proposed as a viable solution for high-speed indoor short-range wireless communication systems because of its robustness to severe multi-path and multi-user conditions, low cost, and low power implementation. This paper proposes and analyzes a multi-user time-hopping system in which symbols are sent multiple times. A single-user receiver structure is proposed in which collisions with interfering users are discarded. Two formulas for the probability of error are derived. The first formula is a finite sum in which the number of terms grows with the number of symbol repetitions. The second formula is an integral that is well-suited to Chebyshev–Gauss quadrature as well as for allowing the derivation of bounds on the probability of error. Asymptotic formulas and upper bounds for the error probability are derived by letting various system parameters go to infinity. We evaluate the bounds numerically for some reasonable parameters and study their interplay with other parameters of interest.**

I. Introduction

Time-hopping combined with pulse position modulation was the original proposal for ultra-wideband (UWB) systems [9], [6]. In digital communications (such as ultra-wideband) design, the theoretical expression representing the error probability, P_e, is often so complicated that it is impractical to use in error probability analysis. Often it is easier to work with a simpler expression, representing an upper bound for P_e [3]. In this paper we study the average error probability of a time-hopping (TH) UWB system in a dense multi-user environment at a relatively low SNR. This paper develops a simple formula involving some parameters of a wideband communication system such as the number of slots, the number of transmission frames, and the number of users. Upper bounds and two exact formulas for the average bit-error probability (BEP) for a time-hopping scheme combined with spread spectrum (pulse position modulation) are derived. Three bounds on the error probability based on the central limit theorem (CLT), Stirling's approximation of the binomial, and the Chernoff bound are also developed and compared. We evaluate the bounds numerically for some reasonable parameters and study its interplay with other parameters of interest.

The organization of the paper is as follows. In Section II, a single-user receiver structure is presented and its signal processing described. An exact formula for the average probability of error of the time-hopping UWB system is derived.

Section III deals with the asymptotic analysis of the average probability of error. Asymptotic formulas are derived by letting various system parameters go to infinity. Upper bounds on the BEP of the system using established mathematical results are derived in this section. Another exact formula for the BEP of the system is derived in this section.

Section IV presents a comparison of the bounds and formulae developed throughout the paper. With reasonable numerical assumptions on the parameters of the system, the bounds and the BEP formulas are compared.

Section V contains our conclusions.

II. Average Probability of Error in a Time-Hopping PPM System with Collisions

A. Time-Hopping PPM System Model

In a time-hopping system, a user conveys a message by sending several delayed copies of a basic pulse $s(t)$. The pulse s is chosen from a finite set of signals $\mathcal{S}$, where each signal in $\mathcal{S}$ has duration at most T_c. The transmission of copies of s takes place over a sequence of N_f "frames," each of duration T_f, where $N_cT_c \le T_f$ for some positive integer N_c. The inequality accounts for a small portion which serves as the guard time. Moreover, the ratio N_cT_c/T_f indicates the fraction of the frame time T_f over which time-hopping (TH) is allowed.

During the ℓth frame, the pulse is transmitted during "chip" $c_\ell \in \{0, \dots, N_c - 1\}$. We call $\{c_0, \dots, c_{N_f-1}\}$ the time-hopping sequence. The total transmitted waveform is

$$x(t) := \sum_{\ell=0}^{N_f-1} s(t - c_\ell T_c - \ell T_f). \qquad (1)$$

B. The Multi-user Environment

In general, there are N_u users transmitting at the same time. However, each user employs a different time-hoping sequence so that in most frames, no two users transmit during the same chip. Let $\mathcal{G} := \{\ell :$ in frame ℓ, no other user transmits during chip $c_\ell\}$. Then $G := |\mathcal{G}|$ is the number of "good" frames; i.e., the number of frames in which no other user transmits during chip c_ℓ of the desired user.

C. A Single-User Receiver Structure

Assuming that the receiver knows the time-hopping sequence $\{c_\ell\}$ of the desired user as well as the good frames $\mathcal{G}$, the receiver looks only at the good frames; i.e., the receiver ignores the frames in which collisions occur. For example, if there are 4 such good frames, the received waveform is depicted in Fig. 1. Then for each $\hat{s} \in \mathcal{S}$, the receiver correlates the incoming waveform with

$$\hat{x}(t) := \sum_{\ell' \in \mathcal{G}} \hat{s}(t - c_{\ell'} T_c - \ell' T_f).$$

Now observe that

$$\begin{aligned}\int x(t)\hat{x}(t)^* dt &= \sum_{\ell=0}^{N_f-1} \sum_{\ell' \in \mathcal{G}} \int s(t - c_\ell T_c - \ell T_f) \times \\ &\qquad \hat{s}(t - c_{\ell'} T_c - \ell' T_f)^* dt.\end{aligned}$$

Since s and $\hat{s}$ are of duration at most T_c, the integral on the right is zero unless $\ell = \ell'$. This can happen only for $\ell \in \mathcal{G}$. Hence,

$$\begin{aligned}\int x(t)\hat{x}(t)^* dt &= \sum_{\ell \in \mathcal{G}} \int s(t - c_\ell T_c - \ell T_f) \times \\ &\qquad \hat{s}(t - c_\ell T_c - \ell T_f)^* dt \\ &= G \int s(t)\hat{s}(t)^* dt.\end{aligned}$$

In other words, from the receiver's point of view, the transmitter might as well have sent the signal $\sum_{\ell \in \mathcal{G}} s(t - c_\ell T_c - \ell T_f)$ instead of $x(t)$ in (1).

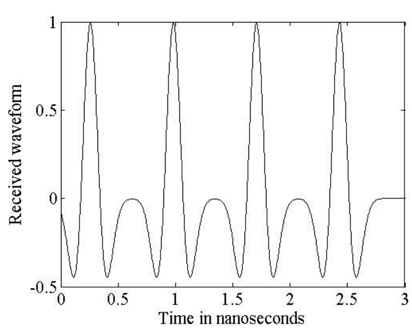

Fig. 1. *Received waveform sequence when there are 4 such good frames.*

In the case of binary signaling, say $\mathcal{S} = \{s_0, s_1\}$, the probability of error for an additive white gaussian noise (AWGN) channel depends only on the distance between

$$\sum_{\ell \in \mathcal{G}} s_1(t - c_\ell T_c - \ell T_f) \text{ and } \sum_{\ell \in \mathcal{G}} s_0(t - c_\ell T_c - \ell T_f).$$

It is easily seen that this distance is equal to $\sqrt{G}$ times the distance between s_0 and s_1. In the case of orthogonal signals s_0 and s_1 of equal energy ξ, the probability of error conditioned on $G = g$ good frames is

$$h(g) := Q\Big(\sqrt{g\xi/\sigma_n^2}\Big), \tag{2}$$

where σ_n^2 is the AWGN power spectral density, and $Q(x) := \int_x^\infty e^{-t^2/2}/\sqrt{2\pi} dt$ is the complementary cumulative distribution function of a standard normal random variable. Ultimately, we are interested in

$$P_e := \mathsf{E}[h(G)] = \mathsf{E}\Big[Q\Big(\sqrt{G\xi/\sigma_n^2}\Big)\Big] \tag{3}$$

computed under the distribution of G, the number of good frames. If no collisions occur in the ℓth frame with a probability p, and if the occurrences of collisions in different frames are independent and identically distributed, then G ~binomial(N_f, p) as discussed next. Note that p depends on the number of chips N_c as well as the number of users N_u.

D. Probability of Good Frames

We define a collision to be the situation when two waveforms belonging to different users occupy the same slot in one frame. Collisions result in the erasure of the desired signal waveform. By construction, the collisions with a user in different slots occur independently.

Proposition 1: As a function of the number of chips N_c and the number of users N_u,

$$p = (1 - 1/N_c)^{N_u - 1}. \tag{4}$$

Proof. The probability of no collision, p, is given by

$$\begin{aligned} p &= P\Big\{\Big(\text{no collision with user } 1\Big) \cap \\ &\qquad \cdots \cap \Big(\text{no collision with user } N_u\Big)\Big\} \\ &= P\Big(\text{no collision with user } 1\Big) \cdot \\ &\qquad \cdots P\Big(\text{no collision with user } N_u\Big) \\ &= \Big(1 - \frac{1}{N_c}\Big) \cdot \Big(1 - \frac{1}{N_c}\Big) \cdots \Big(1 - \frac{1}{N_c}\Big) \\ &= \Big(1 - \frac{1}{N_c}\Big)^{N_u - 1}. \end{aligned}$$

Suppose that the number of users N_u is some fraction λ of the number of chips N_c; i.e., $N_u = \lambda N_c$. Then as N_c becomes large, or equivalently, as N_u becomes large,

$$p = (1 - \lambda/N_u)^{N_u - 1} = \frac{(1 - \lambda/N_u)^{N_u}}{1 - \lambda/N_u} \to e^{-\lambda}. \tag{5}$$

III. Analysis of P_e

A. Asymptotic Analysis

We are interested in

$$P_e = \mathsf{E}[h(G)] \;=\; \mathsf{E}\Big[Q\Big(\sqrt{G\xi/\sigma_n^2}\Big)\Big].$$

Since $G \sim$binomial(N_f, p), we can write

$$\begin{aligned} P_e = \mathsf{E}[h(G)] \;&=\; \mathsf{E}\Big[Q\Big(\sqrt{G\xi/\sigma_n^2}\Big)\Big] \\ &=\; \sum_{g=0}^{N_f} Q\Big(\sqrt{g\xi/\sigma_n^2}\Big)\mathsf{P}(G=g) \\ &=\; \sum_{g=0}^{N_f} Q\Big(\sqrt{g\xi/\sigma_n^2}\Big) \times \\ &\qquad \binom{N_f}{g} p^g(1-p)^{N_f-g}. \end{aligned} \tag{6}$$

From this and (4), we can analyze P_e as a function of N_c, N_u, and N_f. To this end, observe that $h(g)$ in (2) is a bounded, continuous function on $[0,\infty)$. Hence, if G_n converges in distribution to a real-valued random variable G, then $\mathsf{E}[h(G_n)] \to \mathsf{E}[h(G)]$ [2].

Proposition 2: As a function of the number of users N_u,

$$\lim_{N_u\to\infty} P_e \;=\; \mathsf{E}\Big[Q\Big(\sqrt{0\xi/\sigma_n^2}\Big)\Big] \;=\; 1/2. \tag{7}$$

Proof. As the number of users $N_u \to \infty$, we are certain to have collisions in every frame; i.e., $p \to 0$. In this case, $G \sim$ binomial(N_f, p) converges in distribution to the constant random variable 0, and so $\mathsf{E}[h(G)] \to \mathsf{E}[h(0)] = 1/2$ as claimed.

Proposition 3: As a function of the number of chips N_c,

$$\lim_{N_c\to\infty} P_e \;=\; \mathsf{E}\Big[Q\Big(\sqrt{G\xi/\sigma_n^2}\Big)\Big] \;=\; Q\Big(\sqrt{N_f\xi/\sigma_n^2}\Big). \tag{8}$$

Proof. As $N_c \to \infty$, $p \to 1$. In this case, $G \sim$ binomial(N_f, p) converges in distribution to the constant random variable N_f, and so $\mathsf{E}[h(G)] \to \mathsf{E}[h(N_f)]$ as claimed.

Proposition 4: As a function of the number of frames N_f,

$$\lim_{N_f\to\infty} P_e \to 0. \tag{9}$$

Proof. The idea of this proof is that as $N_f \to \infty$, the binomial(N_f, p) random variable G converges in distribution to the constant random variable ∞. However since we are not used to working with random variables that are not finite real valued, we must prove that in this case $\mathsf{E}[h(G)] \to \mathsf{E}[h(\infty)]$ as we would like. We proceed as follows. Since $h(g) \to 0$ as $g \to \infty$, given any $\varepsilon > 0$, there is a $0 < \delta < \infty$ such that $h(g) < \varepsilon$ for all $g > \delta$. Further, since $h(g) \le 1/2$, we can write

$$\begin{aligned} \mathsf{E}[h(G)] \;&\le\; \mathsf{E}[h(G)\{I_{[0,\delta]}(G) + I_{(\delta,\infty)}(G)\}] \\ &\le\; (1/2)\mathsf{P}(G\le\delta) + \varepsilon\mathsf{P}(G>\delta) \\ &<\; (1/2)\mathsf{P}(G\le\delta) + \varepsilon. \end{aligned}$$

Next,

$$\begin{aligned} \mathsf{P}(G\le\delta) \;&=\; \sum_{g=0}^{\lfloor\delta\rfloor} \binom{N_f}{g}\Big(\frac{p}{1-p}\Big)^g(1-p)^{N_f} \\ &\le\; \sum_{g=0}^{\lfloor\delta\rfloor} \frac{N_f^g}{g!}\Big(\frac{p}{1-p}\Big)^g(1-p)^{N_f}. \end{aligned}$$

Writing,

$$N_f^g(1-p)^{N_f} = \exp[g\ln N_f + N_f\ln(1-p)],$$

we see that $\lim_{N_f\to\infty} \mathsf{P}(G\le\delta) \to 0$. Hence,

$$\overline{\lim_{N_f\to\infty}}\, \mathsf{E}[h(G)] \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{N_f\to\infty} \mathsf{E}[h(G)] = 0$ and the proposition is proved.

The limits established in the three propositions are illustrated in Figs. 2–4 in Section IV.

B. Analysis of P_e Using Craig's Formula

Although P_e can be computed directly from (6), we can also give a simple upper bound and an integral formula that avoids computing factorials.

We begin with Craig's formula [4],

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(\frac{-x^2}{2\sin^2 t}\right)dt. \tag{10}$$

This immediately gives the inequality

$$Q(x) \le \frac{1}{2}e^{-x^2/2}. \tag{11}$$

In both formulas, set $x = \sqrt{G\xi/\sigma_n^2}$ and take expectations. This yields both

$$P_e = \frac{1}{\pi}\int_0^{\pi/2} \mathsf{E}\left[\exp\left(\frac{-G\xi/\sigma_n^2}{2\sin^2 t}\right)\right]dt$$

and

$$P_e \le \frac{1}{2}\mathsf{E}[e^{-G\xi/2\sigma_n^2}]. \tag{12}$$

Now, since $G \sim$binomial(N_f, p), its moment generating function is

$$M(\theta) := \mathsf{E}[e^{\theta G}] = [(1-p) + pe^{\theta}]^{N_f}. \tag{13}$$

We thus have

$$P_e = \frac{1}{\pi}\int_0^{\pi/2} M\left(\frac{-\xi/\sigma_n^2}{2\sin^2 t}\right)dt \tag{14}$$

and

$$P_e \le \frac{1}{2}M(-\xi/2\sigma_n^2) = \frac{1}{2}[(1-p) + pe^{-\xi/2\sigma_n^2}]^{N_f}. \tag{15}$$

To conclude, we make the change of variable $y = \sin t$ in (14) to get

$$\begin{aligned} P_e \;&=\; \frac{1}{\pi}\int_0^1 M\Big(\frac{-\xi/\sigma_n^2}{2y^2}\Big)\frac{dy}{\sqrt{1-y^2}} \\ &=\; \frac{1}{2\pi}\int_{-1}^1 M\Big(\frac{-\xi/\sigma_n^2}{2y^2}\Big)\frac{dy}{\sqrt{1-y^2}}, \end{aligned}$$

which is ideally suited to Chebyshev–Gauss quadrature [7, p. 889, formula 25.4.38]; i.e.,

$$P_e \approx \frac{\pi}{K}\sum_{k=1}^{K} M\biggl(\frac{-\xi/\sigma_n^2}{2y_k^2}\biggr), \quad y_k := \cos\biggl[\frac{(2k-1)\pi}{2K}\biggr]. \tag{16}$$

When K is even, we can exploit symmetry to reduce computation by doubling the sum from 1 to $K/2$. A similar reduction can be obtained when K is odd, though care must be taken to use the limiting value $M = (1-p)^{N_f}$ when $y_k = 0$.

C. Analysis of P_e Using the Poisson Approximation

Recall that (15) was obtained by computing the right-hand side of (12) using the true binomial distribution of G. Here we approximate the right-hand side of (12) by treating G as Poisson with parameter pN_f. In this case, the right-hand side of (12) is expressed in terms of the Poisson moment generating function $\exp[pN_f(e^\theta - 1)]$. Thus,

$$P_e \approx \frac{1}{2}\exp[pN_f(e^{-\xi/2\sigma_n^2} - 1)]. \tag{17}$$

Note that this is also immediate from (15) if we take (13) and note that for large N_f with pN_f approximately constant,

$$[(1-p)+pe^\theta]^{N_f} = \biggl[1+\frac{pN_f(e^\theta-1)}{N_f}\biggr]^{N_f} \sim e^{pN_f(e^\theta-1)}. \tag{18}$$

D. Analysis of P_e Using the Central Limit Theorem

In the previous subsection, we approximated the binomial G in (12) with a Poisson. Here we use the central limit theorem to approximate the binomial G in (12) as a Gaussian with mean $m := \mathsf{E}[G] = N_f p$ and variance $\sigma^2 := N_f p(1-p)$. In this case, the right-hand side of (12) is expressed in terms of the Gaussian moment generating function $e^{\theta m + \theta^2\sigma^2/2}$. We thus have the approximation

$$P_e \approx \frac{1}{2} e^{\theta p N_f + \theta^2 N_f p(1-p)}\bigg|_{\theta = -\xi/2\sigma_n^2}. \tag{19}$$

E. Small SNR

Proposition 5: As the signal-to-noise ratio ξ/σ_n^2 goes to zero, (15), (17), and (19) are all close to

$$\frac{1}{2}\exp[-pN_f\xi/2\sigma_n^2]. \tag{20}$$

Proof. In (15), put $x = \xi/2\sigma_n^2$ and use the approximation $e^{-x} \approx 1 - x$ to write

$$\begin{aligned} [(1-p)+pe^{-x}]^{N_f} &\approx [(1-p)+p(1-x)]^{N_f} \\ &= [1-px]^{N_f} \\ &= \exp[N_f \ln(1-px)] \\ &\approx \exp[-pN_f x], \end{aligned}$$

where the last step uses the approximation $\ln t \approx t - 1$ for $t \approx 1$. In (17), use $e^{-x} - 1 \approx -x$ to write $\exp[pN_f(e^{-x}-1)] \approx \exp[-pN_f x]$. In (19), observe that if the signal-to-noise ratio is small, then so is θ. For small θ, we use the approximation $\theta^2 \approx 0$. Then

$$e^{\theta p N_f + \theta^2 N_f p(1-p)} \approx e^{\theta p N_f}.$$

Since $\theta = -x$, the right-hand side is just $e^{-pN_f x}$ as required.

IV. Numerical Results

This section compares the various formulae computed in the paper. Fig. 2 simulates the error probability, P_e versus number of users, N_u. Observe that while $N_u \to \infty$, the average error probability curves converge to the asymptotic value, zero. Intuitively, when the number of users in the system increases, the interference (a collision in every frame) will increase and this will impact the error probability. The number of chips, N_c, and number of frames, N_f, are 100 and 50 respectively. SNR is chosen to be 10 dB for this plot.

In Fig. 3, error probability, P_e, is plotted versus number of chips, N_c as a function of N_u. Observe that while N_u increases and $N_c \to \infty$, the average error probability curves converge to the asymptotic value. When the number of chips N_c becomes very large, then no collisions occur in the ℓth frame with a probability 1. Here N_f is chosen to be 10 and SNR is set at 10 dB.

In Fig. 4, the average error probability, P_e, is plotted versus number of frames, N_f. N_c is chosen to be 100. Observe that as the number of transmitted frames increases P_e drops to its asymptotic value, 0. As the number of users in the system increases, the error probability takes a longer time to reach its asymptotic value, 0. SNR was chosen to be 10 dB for this plot.

Error probability, P_e, versus number of frames, N_f, is plotted in Fig. 5. The average probability of error (6), is plotted against the approximate formulas (upper bounds on bit-error probability) using Craig's formula (10), Poisson approximation (17), and central limit theorem (19). For this plot, N_c, N_f, and N_u were chosen to be 100, 25, and 100. The SNR is good at 30 dB. The bound based on the Craig's formula is more tight compared with the other two bounds, as a function of the number of frames, N_f.

Error probability, P_e, versus p is plotted in Fig. 6. For this plot, N_c, N_f, and N_u were chosen to be 50, 10, and 10000. The SNR was chosen to be 25 dB. From these plots, it can be inferred that the bounds are quite good for lower values of p. For lower values of p, among the three bounds, the one based on Craig's formula serves as the tightest bound.

In Fig. 7, we show that the three upper bounds on the error probability derived in the paper converge for low values of SNR.

V. Conclusions

Two closed-form expressions for the bit-error probability of an ultra-wideband system, without using computationally intensive methods have been developed

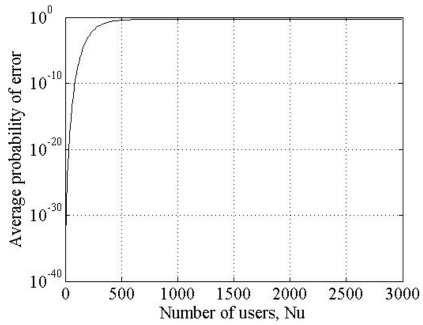

Fig. 2. *Error probability, P_e versus number of users, N_u.*

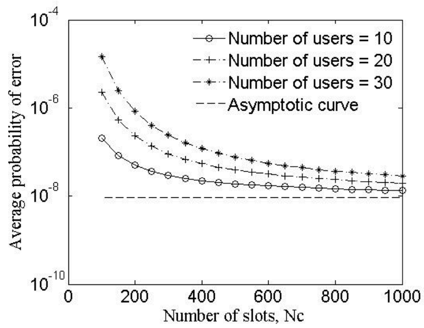

Fig. 3. *Error probability, P_e versus number of chips, N_c as a function of N_u.*

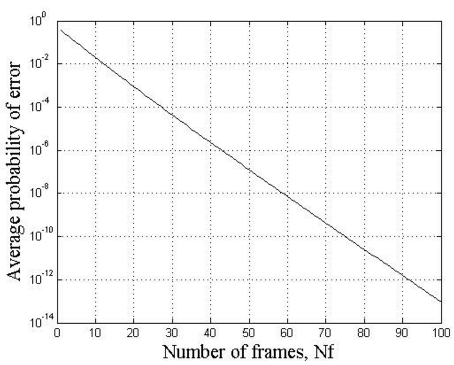

Fig. 4. *The average error probability, P_e is plotted versus number of frames, N_f. N_u was chosen to be 60.*

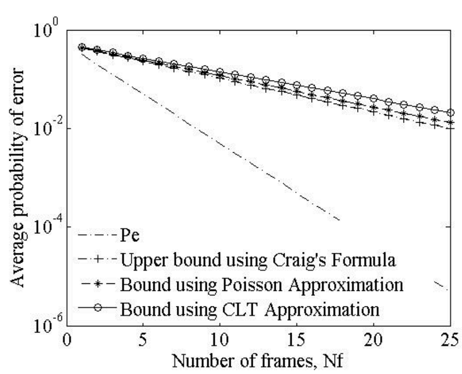

Fig. 5. *Error probability, P_e versus N_f.*

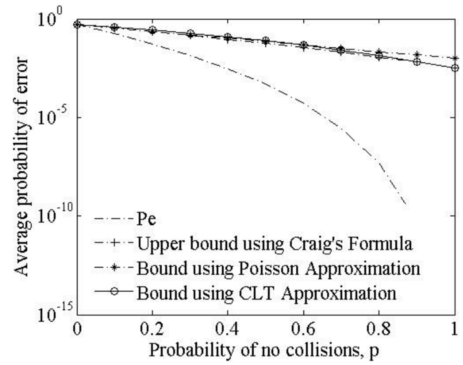

Fig. 6. *Error probability, P_e versus p.*

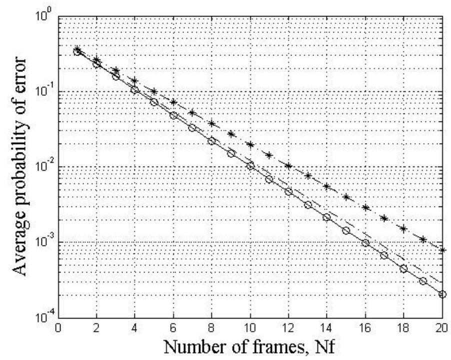

Fig. 7. *The average error probability, P_e is plotted versus number of frames, N_f. SNR is chosen to be $= 1$ dB.*

in this paper. Asymptotic formulas have been derived by letting various system parameters go to infinity and using known mathematical formulae. Easy-to-use upper bounds for error probability analysis have been developed in this paper. The bound using Craig's formula serves as the tightest bound (among the three bounds) in a dense multi-user environment at reasonable SNR values. However, at low SNR values, the three bounds are quite similar. Hence, our method will be a good starting point for researchers needing to compute the average bit error probability in the course of the analysis and design of UWB systems.

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