ECE 330 Supplement on Signals and Systems

1. Signals

A **signal** or **waveform** is a collection of numbers indexed by time. For a discrete-time waveform, we have

$$x \;=\; \{x[n], n = 0, \pm 1, \pm 2, \ldots\}.$$

Here x is the name of the waveform, and for a fixed integer n, $x[n]$ is the value of the waveform x at time n. Similarly, for a continuous-time waveform we have

$$x \;=\; \{x(t), -\infty < t < \infty\}.$$

Again, x is the name of the waveform, and for fixed t, $x(t)$ is the value of the waveform at time t.

Two continuous-time waveforms, say $x = \{x(t), -\infty < t < \infty\}$ and $y = \{y(t), -\infty < t < \infty\}$ are **equal** if and only if

$$x(t) \;=\; y(t) \quad \text{for all times } t.$$

Similarly, two discrete-time waveforms x and y are **equal** if and only if

$$x[n] \;=\; y[n] \quad \text{for all integers } n.$$

A continuous-time waveform x is **bounded** if there is a positive, finite constant K such that

$$|x(t)| \;\le\; K \quad \text{for all times } t.$$

A similar definition holds for discrete-time waveforms.

Given a signal x and a fixed time τ, we define a new waveform x_τ by

$$x_\tau(t) \;:=\; x(t-\tau), \quad -\infty < t < \infty.$$

If $\tau > 0$, x_τ is a **delayed** version of x. If $\tau < 0$, x_τ is an **advanced** version of x. In discrete-time, for fixed m, we define x_m by

$$x_m[n] \;:=\; x[n-m], \quad n = 0, \pm 1, \pm 2, \ldots.$$

2. Systems

A **system** is a function, call it A, that takes as input a waveform x and assigns a corresponding output waveform denoted by Ax. The value of the output waveform at time t is denoted by $(Ax)(t)$ in the continuous-time case. In the discrete-time case, the value of the output waveform at time n is denoted by $(Ax)[n]$.

2.1. System Properties

2.1.1. Memory

A system A is **memoryless** if the value $(Ax)(t)$ can be computed without using any values of $x(s)$ for $s \neq t$. In other words, the output at time t, $(Ax)(t)$, depends at most only on the input signal value at time t, $x(t)$. If a system is not memoryless, we say that it has **memory**.

2.1.2. Causality

A system A is **causal** if the value of $(Ax)(t)$ can be computed without using any values of $x(s)$ for $s > t$. In other words, the output at time t, $(Ax)(t)$, depends at most on the input signal values $x(s)$ at times $s \le t$.

A system that is not causal cannot operate in real time because to compute $(Ax)(t)$ we need to know values of $x(s)$ at future times $s > t$.

2.1.3. Stability

A system A is **stable** if whenever a bounded input waveform x is applied, the corresponding output waveform is also bounded (usually by a different constant). This property is sometimes called bounded input bounded output (BIBO) stability. A system that is not stable is called **unstable**.

It is not possible to build an unstable system because it would have to be able to output an infinite amount of energy.

2.1.4. Time Invariance

A system A is **time invariant** (TI) if for every input signal x and every delay/advance τ,

$$(Ax_\tau)(t) \;=\; (Ax)(t-\tau) \quad \text{for all times } t.$$

In other words, the response to the delayed input x_τ is found by delaying the response to x by τ.

2.1.5. Linearity

We first introduce two preliminary concepts. A system A is **additive** if for every pair of waveforms x_1 and x_2 (here the subscripts do not indicate delays),

$$A(x_1 + x_2) \;=\; Ax_1 + Ax_2.$$

Note that the above formula involves the equality between two signals; i.e., the formula is shorthand for

$$\big(A(x_1 + x_2)\big)(t) \;=\; (Ax_1)(t) + (Ax_2)(t) \quad \text{for all times } t.$$

A system A is **homogeneous** if for every waveform x and every number λ,

$$A(\lambda x) \;=\; \lambda(Ax).$$

Again, this formula is shorthand for

$$\big(A(\lambda x)\big)(t) \;=\; \lambda(Ax)(t) \quad \text{for all times } t.$$

We now define a system A to be **linear** if it is both additive and homogeneous. This is equivalent to the property that for every pair of waveforms x_1 and x_2 and for every pair of numbers λ_1 and λ_2,

$$A(\lambda_1 x_1 + \lambda_2 x_2) \;=\; \lambda_1(Ax_1) + \lambda_2(Ax_2).$$

Again, this is shorthand for

$$\big(A(\lambda_1 x_1 + \lambda_2 x_2)\big)(t) = \lambda_1 (Ax_1)(t) + \lambda_2 (Ax_2)(t) \quad \text{for all times } t.$$

Linearity has several implications. Focusing on the homogeneity equation $A(\lambda x) = \lambda(Ax)$, observe that taking $\lambda = 0$ yields $A(0) = 0$; i.e.,

> **The response to the zero waveform must be the zero waveform.**

Taking $\lambda = -1$ yields $A(-x) = -(Ax)$; i.e.,

> **The response to –*x* must be the negative of the response to *x*.**

Taking $\lambda = 2$ yields $A(2x) = 2(Ax)$; i.e.,

> **If we double the input, the output must double.**

Other such implications can be derived. ***The point here is that if any such implication fails, the system cannot be linear.***