ECE 330 Supplement on Signals and Systems

1. Signals

A signal or waveform is a collection of numbers indexed by time. For a discrete-time waveform, we have

$$x = \{x[n], n = 0, \pm 1, \pm 2, \ldots \}.$$ 

Here $x$ is the name of the waveform, and for a fixed integer $n$, $x[n]$ is the value of the waveform $x$ at time $n$. Similarly, for a continuous-time waveform we have

$$x = \{x(t), -\infty < t < \infty \}.$$ 

Again, $x$ is the name of the waveform, and for fixed $t$, $x(t)$ is the value of the waveform at time $t$.

Two continuous-time waveforms, say $x = \{x(t), -\infty < t < \infty \}$ and $y = \{y(t), -\infty < t < \infty \}$ are equal if and only if

$$x(t) = y(t) \quad \text{for all times } t.$$ 

Similarly, two discrete-time waveforms $x$ and $y$ are equal if and only if

$$x[n] = y[n] \quad \text{for all integers } n.$$ 

A continuous-time waveform $x$ is bounded if there is a positive, finite constant $K$ such that

$$|x(t)| \leq K \quad \text{for all times } t.$$ 

A similar definition holds for discrete-time waveforms.

Given a signal $x$ and a fixed time $\tau$, we define a new waveform $x_\tau$ by

$$x_\tau(t) := x(t - \tau), \quad -\infty < t < \infty.$$ 

If $\tau > 0$, $x_\tau$ is a delayed version of $x$. If $\tau < 0$, $x_\tau$ is an advanced version of $x$. In discrete-time, for fixed $m$, we define $x_m$ by

$$x_m[n] := x[n - m], \quad n = 0, \pm 1, \pm 2, \ldots.$$ 

2. Systems

A system is a function, call it $A$, that takes as input a waveform $x$ and assigns a corresponding output waveform denoted by $Ax$. The value of the output waveform at time $t$ is denoted by $(Ax)(t)$ in the continuous-time case. In the discrete-time case, the value of the output waveform at time $n$ is denoted by $(Ax)[n]$.

2.1. System Properties

2.1.1. Memory

A system $A$ is memoryless if the value $(Ax)(t)$ can be computed without using any values of $x(s)$ for $s \neq t$. In other words, the output at time $t$, $(Ax)(t)$, depends at most only on the input signal value at time $t$, $x(t)$. If a system is not memoryless, we say that it has memory.

2.1.2. Causality

A system $A$ is causal if the value of $(Ax)(t)$ can be computed without using any values of $x(s)$ for $s > t$. In other words, the output at time $t$, $(Ax)(t)$, depends at most only on the input signal values $x(s)$ at times $s \leq t$.

A system that is not causal cannot operate in real time because to compute $(Ax)(t)$ we need to know values of $x(s)$ at future times $s > t$.

2.1.3. Stability

A system $A$ is stable if whenever a bounded input waveform $x$ is applied, the corresponding output waveform is also bounded (usually by a different constant). This property is sometimes called bounded input bounded output (BIBO) stability. A system that is not stable is called unstable.

It is not possible to build an unstable system because it would have to be able to output an infinite amount of energy.

2.1.4. Time Invariance

A system $A$ is time invariant (TI) if for every input signal $x$ and every delay/advance $\tau$,

$$(Ax_\tau)(t) = (Ax)(t - \tau) \quad \text{for all times } t.$$ 

In other words, the response to the delayed input $x_\tau$ is found by delaying the response to $x$ by $\tau$.

2.1.5. Linearity

We first introduce two preliminary concepts. A system $A$ is additive if for every pair of waveforms $x_1$ and $x_2$ (here the subscripts do not indicate delays),

$$A(x_1 + x_2) = Ax_1 + Ax_2.$$ 

Note that the above formula involves the equality between two signals; i.e., the formula is shorthand for

$$(A(x_1 + x_2))(t) = (Ax_1)(t) + (Ax_2)(t) \quad \text{for all times } t.$$ 

A system $A$ is homogeneous if for every waveform $x$ and every number $\lambda$,

$$A(\lambda x) = \lambda (Ax).$$ 

Again, this formula is shorthand for

$$A(\lambda x)(t) = \lambda (Ax)(t) \quad \text{for all times } t.$$ 

We now define a system $A$ to be linear if it is both additive and homogeneous. This is equivalent to the property that for every pair of waveforms $x_1$ and $x_2$ and for every pair of numbers $\lambda_1$ and $\lambda_2$,

$$A(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 (Ax_1) + \lambda_2 (Ax_2).$$
Again, this is shorthand for

\[ (A(\lambda_1 x_1 + \lambda_2 x_2))(t) = \lambda_1 (Ax_1)(t) + \lambda_2 (Ax_2)(t) \] 

for all times \( t \).

Linearity has several implications. Focusing on the homogeneity equation \( A(\lambda x) = \lambda (Ax) \), observe that taking \( \lambda = 0 \) yields \( A(0) = 0 \); i.e.,

The response to the zero waveform must be the zero waveform.

Taking \( \lambda = -1 \) yields \( A(-x) = -(Ax) \); i.e.,

The response to \(-x\) must be the negative of the response to \( x \).

Taking \( \lambda = 2 \) yields \( A(2x) = 2(Ax) \); i.e.,

If we double the input, the output must double.

Other such implications can be derived. The point here is that if any such implication fails, the system cannot be linear.