Chapter 5, Problem 31 Solution

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\begin{aligned}
\varphi_Y(\nu) &= \mathsf{E}[e^{j\nu Y}] \\
&= \sum_{n=1}^{\infty} \mathsf{E}[e^{j\nu Y}|N=n]\wp(N=n), && \text{by the law of total probability,} \\
&= \sum_{n=1}^{\infty} \mathsf{E}\left[e^{j\nu \sum_{i=1}^{N} X_i}\middle| N=n\right]\wp(N=n) \\
&= \sum_{n=1}^{\infty} \mathsf{E}\left[e^{j\nu \sum_{i=1}^{n} X_i}\middle| N=n\right]\wp(N=n), && \text{by substitution,} \\
&= \sum_{n=1}^{\infty} \mathsf{E}\left[e^{j\nu \sum_{i=1}^{n} X_i}\right]\wp(N=n), && \text{by independence,} \\
&= \sum_{n=1}^{\infty} \left(\prod_{i=1}^{n} \mathsf{E}[e^{j\nu X_i}]\right)\wp(N=n), && \text{by independence,} \\
&= \sum_{n=1}^{\infty} \varphi_X(\nu)^n \wp(N=n) \\
&= G_N(\varphi_X(\nu)).
\end{aligned}
$$