The Poisson Process

1. Counting Processes

A counting process is a random process that counts the number of occurrences of some event during various intervals of time. Below we describe three different applications of counting processes.

When a substance undergoes radioactive decay, we denote by N_t the *random* number of radioactive particles emitted during the time interval $(0, t]$. Similarly, $N_t - N_s$ denotes the number of particles emitted during the interval $(s, t]$.

Another example of a counting process occurs in the analysis of telephone switching systems. Suppose we are monitoring telephone calls arriving at a switching center. Then we can let $N_t - N_s$ denote the random number of calls that arrive during the interval $(s, t]$.

Still a third application of counting processes arises in the analysis of photodetectors. Consider, for example, a fiber-optic network. To convert an optical signal into an electronic signal, the optical fiber output is directed at the surface of a photodetector. It is customary to denote by $N_t - N_s$, the random number of photoelectrons generated in the photodetector during the interval $(s, t]$.

2. The Poisson Process

The Poisson process is a special case of a counting process. It finds wide application in biomedical engineering where it is used in the modeling of radioactive imaging systems. It is also used extensively in the analysis of message traffic in communication networks and in the study of electro-optic devices.

Definition 2.1. A random process $\{X_t\}$ has **independent increments** if for every

$$t_0 < t_1 < \cdots < t_n,$$

the random variables

$$(X_{t_1} - X_{t_0}),\ (X_{t_2} - X_{t_1}),\ \ldots,\ (X_{t_n} - X_{t_{n-1}})$$

are independent.

Suppose that $\{N_t, t \geq 0\}$ is a random process with the following properties:

1. $N_0 \equiv 0$.
2. During a sufficiently short time interval, $\Delta t > 0$, $\wp(N_{t+\Delta t} - N_t = 1) \approx \lambda \Delta t$. By this we mean that
$$\lim_{\Delta t \downarrow 0} \frac{\wp(N_{t+\Delta t} - N_t = 1)}{\Delta t} = \lambda.$$
This property can be interpreted as saying that the probability of observing exactly one particle, telephone call, or photoelectron during a short time interval of length Δt is approximately $\lambda \Delta t$. The parameter λ is called the *rate* or the *intensity* of the process.
3. For sufficiently small $\Delta t > 0$, $\wp(N_{t+\Delta t} - N_t = 0) \approx 1 - \lambda \Delta t$. More precisely,
$$\lim_{\Delta t \downarrow 0} \frac{1 - \wp(N_{t+\Delta t} - N_t = 0)}{\Delta t} = \lambda.$$
By combining this property with the preceding one, we see that during a short time interval of length Δt, either exactly one event occurs, or no event occurs. In other words, during a short time interval, at most one particle, telephone call, or photoelectron will be observed.
4. The process $\{N_t, t \geq 0\}$ has independent increments. This property says that the numbers of events occurring during disjoint time intervals are independent random variables.

We now proceed to derive a formula for $\wp(N_t - N_s = n)$. For $n = 0, 1, \ldots$, let

$$p_n(t) \triangleq \wp(N_t - N_s = n), \qquad t \geq s. \tag{2.1}$$

Note that $p_0(s) = \wp(N_s - N_s = 0) = \wp(0 = 0) = \wp(S) = 1$. Now,

$$\begin{aligned}
&p_n(t + \Delta t) \\
&\quad = \wp(N_{t+\Delta t} - N_s = n) \\
&\quad = \wp\left(\bigcup_{k=0}^{n} \Big[\{N_t - N_s = n - k\} \cap \{N_{t+\Delta t} - N_t = k\} \Big] \right) \\
&\quad = \sum_{k=0}^{n} \wp(N_{t+\Delta t} - N_t = k,\ N_t - N_s = n - k) \\
&\quad = \sum_{k=0}^{n} \wp(N_{t+\Delta t} - N_t = k) p_{n-k}(t),
\end{aligned}$$

where last step follows from property 4 and (2.1). Break the preceding sum into three terms as follows.

$$\begin{aligned}
p_n(t + \Delta t) = {} & \wp(N_{t+\Delta t} - N_t = 0) p_n(t) \\
& + \wp(N_{t+\Delta t} - N_t = 1) p_{n-1}(t) \\
& + \sum_{k=2}^{n} \wp(N_{t+\Delta t} - N_t = k) p_{n-k}(t).
\end{aligned}$$

This enables us to write

$$\begin{aligned}
&p_n(t + \Delta t) - p_n(t) \\
&\qquad = -[1 - \wp(N_{t+\Delta t} - N_t = 0)] p_n(t) \\
&\qquad\quad + \wp(N_{t+\Delta t} - N_t = 1) p_{n-1}(t) \\
&\qquad\quad + \sum_{k=2}^{n} \wp(N_{t+\Delta t} - N_t = k) p_{n-k}(t).
\end{aligned} \tag{2.2}$$

For $n = 0$, only the first term in (2.2) is present, and we can write

$$p_0(t + \Delta t) - p_0(t) = -[1 - \wp(N_{t+\Delta t} - N_t = 0)] p_0(t). \tag{2.3}$$

It then follows that

$$\lim_{\Delta t \downarrow 0} \frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} \;=\; -\lambda p_0(t).$$

In other words, we are left with the first-order differential equation,

$$p_0'(t) \;=\; -\lambda p_0(t), \qquad p_0(s) = 1,$$

whose solution is simply

$$p_0(t) \;=\; e^{-\lambda(t-s)}, \qquad t \geq s.$$

To handle the case $n \geq 2$, note that since

$$\begin{aligned}
&\sum_{k=2}^{n} \wp(N_{t+\Delta t} - N_t = k) p_{n-k}(t) \\
&\quad \leq \sum_{k=2}^{n} \wp(N_{t+\Delta t} - N_t = k) \\
&\quad = \wp(N_{t+\Delta t} - N_t \geq 2) \\
&\quad = 1 - [\, \wp(N_{t+\Delta t} - N_t = 0) + \wp(N_{t+\Delta t} - N_t = 1)\,],
\end{aligned}$$

it follows that

$$\lim_{\Delta t \downarrow 0} \frac{\sum_{k=2}^{n} \wp(N_{t+\Delta t} - N_t = k) p_{n-k}(t)}{\Delta t} \;=\; \lambda - \lambda \;=\; 0.$$

Returning to (2.2), we can now see that for $n = 1$ and for $n \geq 2$,

$$\lim_{\Delta t \downarrow 0} \frac{p_n(t+\Delta t) - p_n(t)}{\Delta t} \;=\; -\lambda p_n(t) + \lambda p_{n-1}(t).$$

This results in the differential-difference equation,

$$p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t), \quad p_0(t) = e^{-\lambda(t-s)}. \tag{2.4}$$

It is easily verified that for $n = 0, 1, \ldots,$

$$p_n(t) \;=\; \frac{[\lambda(t-s)]^n e^{-\lambda(t-s)}}{n!}$$

solves (2.4).

We have now shown that properties 1 through 4 imply that

$$\boxed{\wp(N_t - N_s = n) \;=\; \frac{[\lambda(t-s)]^n e^{-\lambda(t-s)}}{n!}, \quad t \geq s.} \tag{2.5}$$

In fact, by property 4, for any $t_0 < \cdots < t_n$,

$$\begin{aligned}
&\wp((N_{t_1} - N_{t_0}) = k_1, \ldots, (N_{t_n} - N_{t_{n-1}}) = k_n) \\
&\qquad = \prod_{i=1}^{n} \frac{[\lambda(t_i - t_{i-1})]^{k_i} e^{-\lambda(t_i - t_{i-1})}}{k_i!}.
\end{aligned}$$

We can actually do even more by taking $t_0 = 0$ and recalling that $N_0 \equiv 0$. If $0 = \ell_0 \leq \ell_1 \leq \cdots \leq \ell_n$, then

$$\begin{aligned}
&\wp(N_{t_1} = \ell_1, \ldots, N_{t_n} = \ell_n) \\
&\quad = \wp((N_{t_1} - N_0) = \ell_1, \ldots, (N_{t_n} - N_{t_{n-1}}) = (\ell_n - \ell_{n-1})) \\
&\quad = \prod_{i=1}^{n} \frac{[\lambda(t_i - t_{i-1})]^{\ell_i - \ell_{i-1}} e^{-\lambda(t_i - t_{i-1})}}{(\ell_i - \ell_{i-1})!}.
\end{aligned} \tag{2.6}$$

Definition 2.2. A stochastic process, $\{N_t, t \geq 0\}$ is said to be a **Poisson process** if its finite-dimensional probability mass distributions are given by (2.6) and if $\wp(N_0 = 0) = 1$.

Theorem 2.3. *If $\{N_t, t \geq 0\}$ is a Poisson process in the sense of the preceding definition, then the process has independent increments.*

Theorem 2.4. *If a process has independent increments, if $\wp(N_0 = 0) = 1$, and if (2.5) holds, then the process is a Poisson process.*