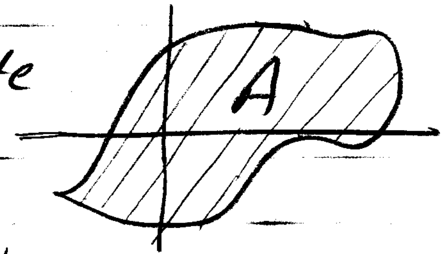
Probabilities for Pairs of RVs

Question 1.

Suppose that when A is a two-dimensional subset of the plane $\mathbb{R}^2$, we can compute $\mathbb{P}((X,Y)\in A)$.

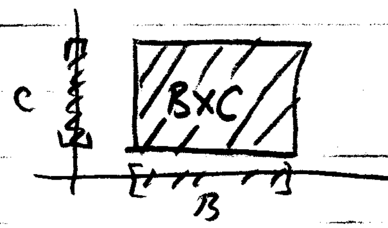
Then how can we compute probabilities that involve only X or Y, say $\mathbb{P}(X\in B)$ or $\mathbb{P}(Y\in C)$, where B and C are one-dimensional sets?



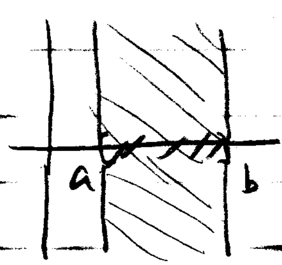
We can use <u>Cartesian products:</u>

$$B\times C := \{(x,y) : x\in B \text{ AND } y\in C\}$$

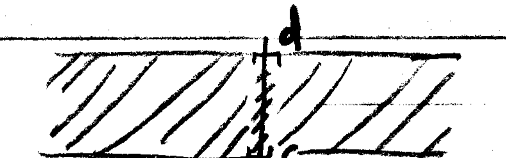
IF B & C ARE INTERVALS THEN $B\times C$ LOOKS LIKE:



In particular $[a,b]\times\mathbb{R}$ is the infinite vertical strip



and $\mathbb{R}\times[c,d]$ is the infinite horizontal strip



We can now see that

$$\mathcal{P}(X \in B) = \mathcal{P}(\{X \in B\} \cap \Omega)$$
$$= \mathcal{P}(\{X \in B\} \cap \{Y \in \mathbb{R}\})$$
$$= \mathcal{P}((X,Y) \in B \times \mathbb{R})$$

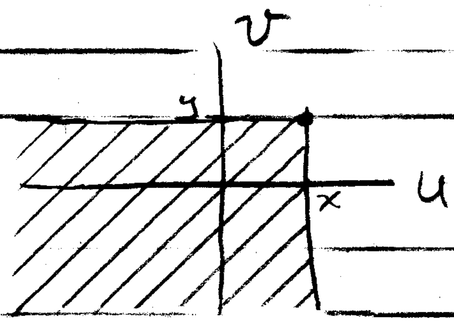
and

$$\mathcal{P}(Y \in C) = \mathcal{P}(\Omega \cap \{Y \in C\})$$
$$= \mathcal{P}(\{X \in \mathbb{R}\} \cap \{Y \in C\})$$
$$= \mathcal{P}((X,Y) \in \mathbb{R} \times C).$$

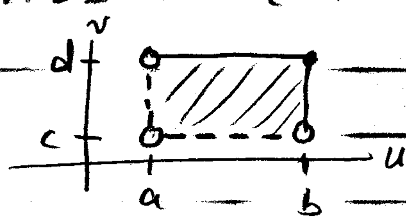
__Question 2__ Now suppose that we can only compute probabilities of the form

$$F_{XY}(x,y) := \mathcal{P}(X \le x, Y \le y)$$ ← joint CDF
$$= \mathcal{P}(X \in (-\infty, x], Y \in (-\infty, y])$$
$$= \mathcal{P}((X,Y) \in (-\infty, x] \times (-\infty, y])$$

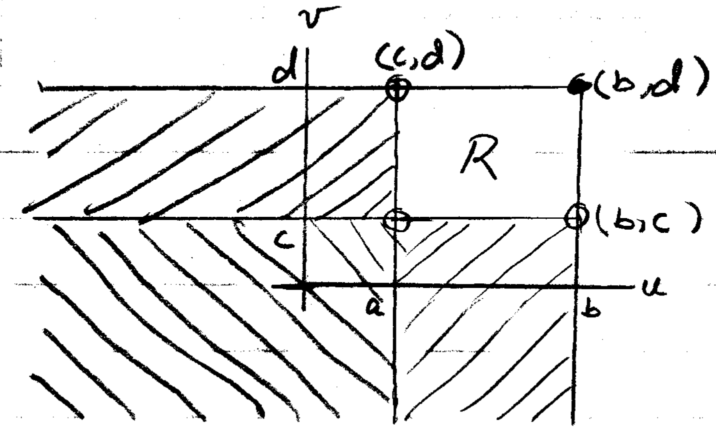
The product set
$$(-\infty, x] \times (-\infty, y]$$
is the "southwest corner":



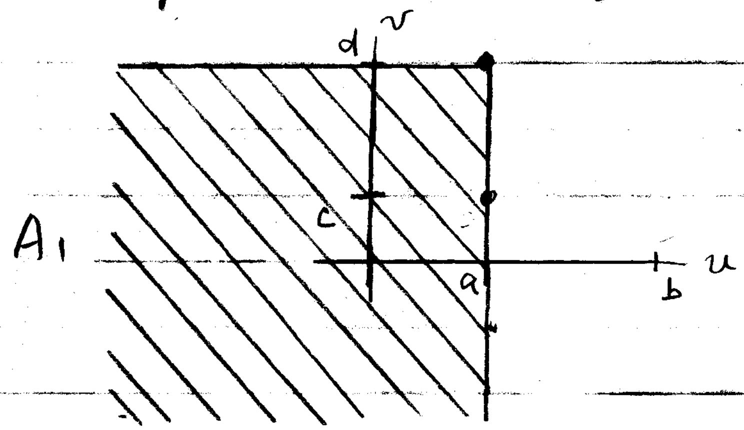
How can we compute the probability that (X,Y) lies in a rectangle, say $(a,b] \times (c,d]$?



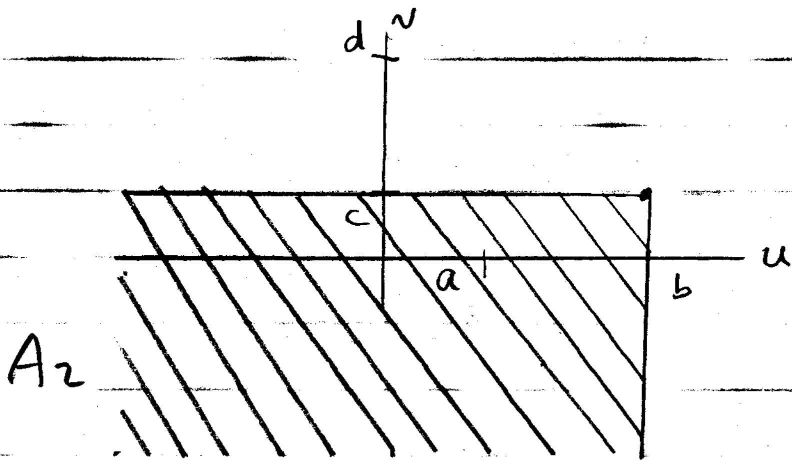
We can find the answer by thinking of the rectangle as a part of a bigger southwest corner:



Let $R := (a,b] \times (c,d]$ denote the desired rectangle.
Let $A_1 := (-\infty,a] \times (-\infty,d]$ denote the southwest corner



Let $A_2 := (-\infty,b] \times (-\infty,c]$ denote the southwest corner



Then the big southwest corner $(-\infty,b] \times (-\infty,d]$ is equal to $R \cup A_1 \cup A_2$

disjoint

So,

$$F_X(b,d) = P\big((X,Y) \in (-\infty, b] \times (-\infty, d]\big)$$

$$= P\big((X,Y) \in R\big) + P\big((X,Y) \in A_1 \cup A_2\big)$$

Now, since A_1 and A_2 are not disjoint, but contain the south west corner $(-\infty, a] \times (-\infty, c]$ in common, we use the inclusion-exclusion formula to write

$$P\big((X,Y) \in A_1 \cup A_2\big) = P\big((X,Y) \in A_1\big) + P\big((X,Y) \in A_2\big) - P\big((X,Y) \in (-\infty, a] \times (-\infty, c]\big)$$

$$= F_{XY}(a,d) + F_{XY}(b,c) - F_{XY}(a,c).$$

So,

$$P\big((X,Y) \in R\big) = F_{XY}(b,d) - F_{XY}(a,d) - F_{XY}(b,c) + F_{XY}(a,c).$$

– the rectangle formula