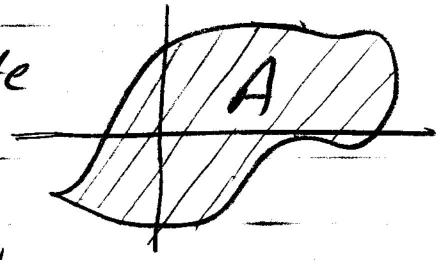
Probabilities for Pairs of RVs

Question 1. Suppose that when A is a two-dimensional subset of the plane $\mathbb{R}^2$, we can compute $\mathbb{P}((X,Y)\in A)$.

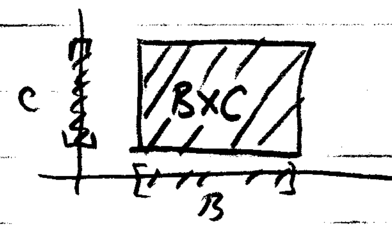
Then how can we compute probabilities that involve only X or Y, say $\mathbb{P}(X\in B)$ or $\mathbb{P}(Y\in C)$, where B and C are one-dimensional sets?



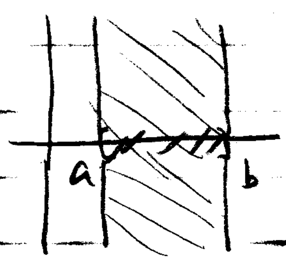
We can use Cartesian products:

$$B\times C := \{(x,y) : x\in B \text{ AND } y\in C\}$$

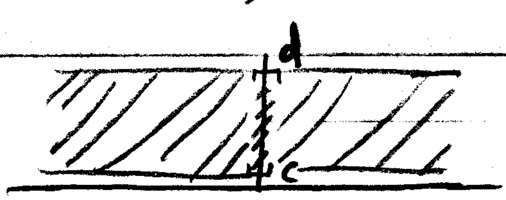
IF B & C ARE INTERVALS THEN $B\times C$ LOOKS LIKE:



In particular $[a,b]\times\mathbb{R}$ is the infinite vertical strip



and $\mathbb{R}\times[c,d]$ is the infinite horizontal strip



We can now see that

$$\mathbb{P}(X \in B) = \mathbb{P}(\{X \in B\} \cap \Omega)$$
$$= \mathbb{P}(\{X \in B\} \cap \{Y \in \mathbb{R}\})$$
$$= \mathbb{P}((X,Y) \in B \times \mathbb{R})$$

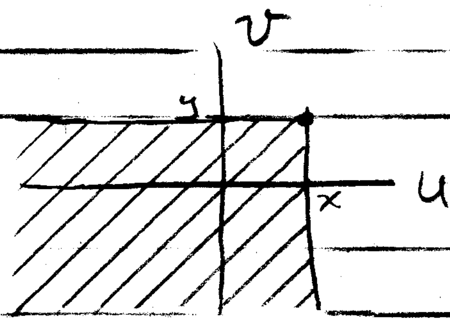
and

$$\mathbb{P}(Y \in C) = \mathbb{P}(\Omega \cap \{Y \in C\})$$
$$= \mathbb{P}(\{X \in \mathbb{R}\} \cap \{Y \in C\})$$
$$= \mathbb{P}((X,Y) \in \mathbb{R} \times C).$$

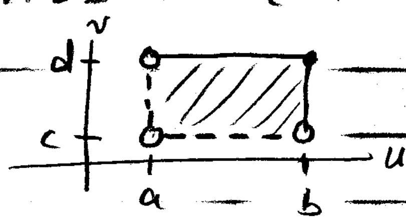
<u>Question 2</u> Now suppose that we can only compute probabilities of the form

$$F_{XY}(x,y) := \mathbb{P}(X \le x, Y \le y) \quad \leftarrow \text{joint CDF}$$
$$= \mathbb{P}(X \in (-\infty, x], Y \in (-\infty, y])$$
$$= \mathbb{P}((X,Y) \in (-\infty, x] \times (-\infty, y])$$

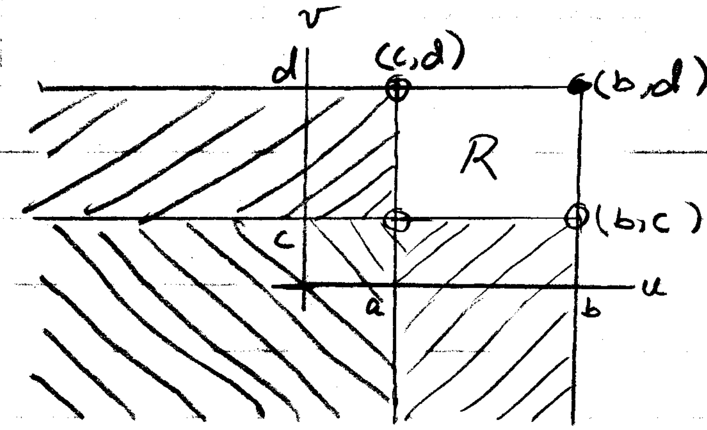
The product set $(-\infty, x] \times (-\infty, y]$ is the "southwest corner":



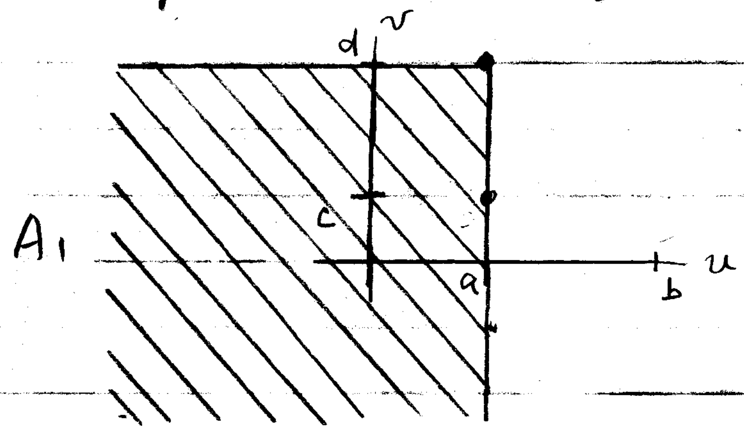
How can we compute the probability that (X,Y) lies in a rectangle, say $(a,b] \times (c,d]$?



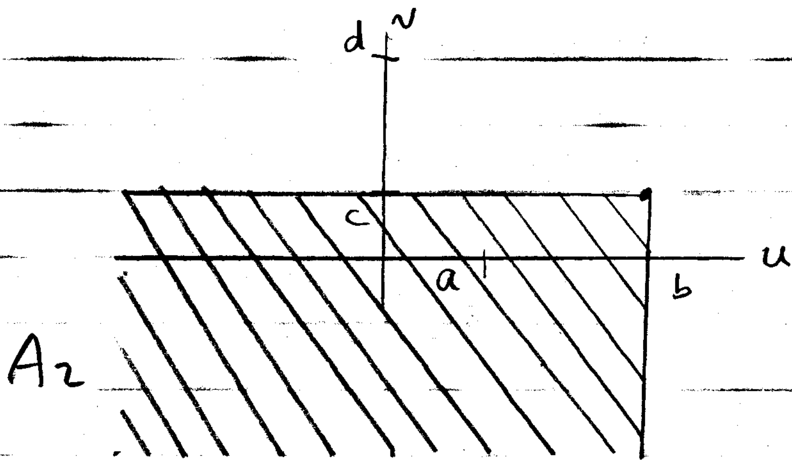
We can find the answer by thinking of the rectangle as a part of a bigger southwest corner:



Let $R := (a,b] \times (c,d]$ denote the desired rectangle.
Let $A_1 := (-\infty, a] \times (-\infty, d]$ denote the southwest corner



Let $A_2 := (-\infty, b] \times (-\infty, c]$ denote the southwast corner



Then the big southwast corner $(-\infty, b] \times (-\infty, d]$
is equal to $\underbrace{R} \cup \underbrace{A_1 \cup A_2}$

disjoint

So,

$$F_X(b,d) = P((X,Y) \in (-\infty, b] \times (-\infty, d])$$

$$= P((X,Y) \in R) + P((X,Y) \in A_1 \cup A_2)$$

Now, since A_1 and A_2 are not disjoint, but contain the south west corner $(-\infty, a] \times (-\infty, c]$ in common, we use the inclusion-exclusion formula to write

$$P((X,Y) \in A_1 \cup A_2) = P((X,Y) \in A_1) + P((X,Y) \in A_2) - P((X,Y) \in (-\infty, a] \times (-\infty, c])$$

$$= F_{XY}(a,d) + F_{XY}(b,c) - F_{XY}(a,c).$$

So,

$$P((X,Y) \in R) = F_{XY}(b,d) - F_{XY}(a,d) - F_{XY}(b,c) + F_{XY}(a,c).$$

– the rectangle formula