The Hilbert Transform

Definition

The **Hilbert transform** of a time function $g(t)$ is is the time function $\hat{g}(t)$ computed as follows. Let $G(f)$ denote the Fourier transform of $g(t)$,

$$G(f) \;:=\; \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}\, dt.$$

Put

$$\hat{G}(f) \;:=\; -j\,\mathrm{sgn}(f)\cdot G(f),$$

where

$$\mathrm{sgn}(f) \;:=\; \begin{cases} 1, & f>0, \\ 0, & f=0, \\ -1, & f<0, \end{cases}$$

Then $\hat{g}(t)$ is defined to be the inverse Fourier transform of $\hat{G}(f)$, i.e.,

$$\hat{g}(t) \;:=\; \int_{-\infty}^{\infty} \hat{G}(f) e^{j2\pi ft}\, df.$$

Properties

A couple of observations are in order. First, note that to compute the Hilbert transform of the Hilbert transform, we compute the inverse Fourier transform of $-j\,\mathrm{sgn}(f)[-j\,\mathrm{sgn}(f)\cdot G(f)] = -G(f)$. Hence, the Hilbert transform of the Hilbert transform of $g(t)$ is $-g(t)$. In symbols,

$$\hat{\hat{g}}(t) = -g(t).$$

Next, it is easy to see that if $g(t)$ is real-valued, then so is $\hat{g}(t)$. First note that $g(t)$ is real if and only if $G(-f) = G(f)^*$. Next, since sgn is odd, $G(-f) = G(f)^*$ if and only if $\hat{G}(-f) = \hat{G}(f)^*$. Hence, $g(t)$ is real if and only if $\hat{g}(t)$ is real.

Example. If $g(t) = \cos(2\pi f_0 t)$, then $G(f) = \frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$. So,

$$\begin{aligned} \hat{G}(f) &= -j\,\mathrm{sgn}(f)\cdot G(f) \\ &= -j\,\mathrm{sgn}(f)\cdot \frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)] \\ &= \frac{1}{2j}[\delta(f-f_0)-\delta(f+f_0)]. \end{aligned}$$

Hence, $\hat{g}(t) = \sin(2\pi f_0 t)$. Since

$$\sin(2\pi f_0 t) \;=\; \cos(2\pi f_0 t - \pi/2),$$

the Hilbert transform is said to implement a 90° phase shift. Writing

$$\cos(2\pi f_0 t - \pi/2) \;=\; \cos\left(2\pi f_0\left[t - \frac{\pi/2}{2\pi f_0}\right]\right),$$

we see that the corresponding time delay is frequency dependent. Similarly, it can be shown that the Hilbert transform of $\sin(2\pi f_0 t)$ is $-\cos(2\pi f_0 t)$.

We also remark that if we put $H(f) = -j\,\mathrm{sgn}(f)$, then $\hat{G}(f) = H(f)G(f)$. Hence, if we let $h(t)$ denote the inverse Fourier transform of $H(f)$, then $\hat{g}(t)$ is given by the convolution integral $\hat{g}(t) = \int_{-\infty}^{\infty} h(t-\tau)g(\tau)\,d\tau$. Since the inverse Fourier transform can be shown to be $h(t) = 1/(\pi t)$, we can write

$$\hat{g}(t) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau}\,d\tau,$$

from which we see once again that if $g(t)$ is real, then so is $\hat{g}(t)$.

Envelopes and Phase

The **pre-envelope** or **analytic signal** of $g(t)$ is the complex-valued function

$$g_+(t) \;:=\; g(t) + j\hat{g}(t), \tag{1}$$

where $\hat{g}(t)$ is the Hilbert transform of $g(t)$. In the frequency domain,

$$G_+(f) \;=\; \begin{cases} 2G(f), & f>0, \\ G(0), & f=0, \\ 0, & f<0. \end{cases}$$

The **envelope** and **angle** of $g_+(t)$ are

$$V(t) \;:=\; |g_+(t)| \quad\text{and}\quad \Theta(t) \;:=\; \arg g_+(t).$$

In other words,

$$g_+(t) \;=\; |g_+(t)|e^{j\arg g_+(t)} \;=\; V(t)e^{j\Theta(t)}.$$

If $g(t)$ is real, then so is $\hat{g}(t)$, and so in this case, $V(t) = \sqrt{g(t)^2+\hat{g}(t)^2}$.

For any convenient frequency f_0, we can define

$$\begin{aligned} g_I(t) &:= g(t)\cos(2\pi f_0 t) + \hat{g}(t)\sin(2\pi f_0 t) \\ g_Q(t) &:= \hat{g}(t)\cos(2\pi f_0 t) - g(t)\sin(2\pi f_0 t). \end{aligned}$$

An easy calculation with simple trigonometric identities shows that

$$g(t) \;=\; g_I(t)\cos(2\pi f_0 t) - g_Q(t)\sin(2\pi f_0 t), \tag{2}$$

where $g_I(t)$ is called the **in-phase** component of $g(t)$, and $g_Q(t)$ is called the **quadrature** component of $g(t)$. A similar calculation shows that

$$\hat{g}(t) \;=\; g_Q(t)\cos(2\pi f_0 t) + g_I(t)\sin(2\pi f_0 t).$$

If $G(f)$ is bandpass, then so is $\hat{G}(f)$; it further follows, using a simple graphical analysis, that $G_I(f)$ and $G_Q(f)$ are lowpass. Hence,

$$\tilde{g}(t) \;:=\; g_I(t) + j g_Q(t)$$

is called the **lowpass representation** of $g(t)$. A simple trig. calculation shows that

$$g_+(t) \;=\; \tilde{g}(t) e^{j2\pi f_0 t}, \tag{3}$$

which explains why $\tilde{g}(t)$ is sometimes called the **complex envelope** of $g(t)$. When $g(t)$ is real, so is $\hat{g}(t)$. In this case, comparing (1) and (3), we have

$$g(t) \;=\; \mathrm{Re}[\tilde{g}(t) e^{j2\pi f_0 t}] \quad \text{and} \quad \hat{g}(t) \;=\; \mathrm{Im}[\tilde{g}(t) e^{j2\pi f_0 t}].$$

In other words, $\tilde{g}(t)$ can be interpreted as a time-varying phasor.

To continue the analysis, write $\tilde{g}(t)$ in polar form as

$$\tilde{g}(t) \;=\; a(t) e^{j\varphi(t)}.$$

Substituting this in (3) yields

$$g_+(t) \;=\; a(t) e^{j[2\pi f_0 t + \varphi(t)]}.$$

Comparing this with (1), we have

$$g(t) \;=\; \mathrm{Re}[a(t) e^{j[2\pi f_0 t + \varphi(t)]}] \;=\; a(t)\cos\bigl(2\pi f_0 t + \varphi(t)\bigr).$$

Thus, $a(t)$ is called the **envelope** of $g(t)$, and $\varphi(t)$ is called the **phase** of $g(t)$. We also have

$$\hat{g}(t) \;=\; \mathrm{Im}[a(t) e^{j[2\pi f_0 t + \varphi(t)]}] \;=\; a(t)\sin\bigl(2\pi f_0 t + \varphi(t)\bigr).$$