# ECE 431

## Fall 2007 Summary for Exam 1

- ***The CTFT and the DTFT.*** Let $x(t)$ be a continuous-time waveform with continuous-time Fourier transform $X(f)$. If $x(t)$ is sampled at the sampling rate $f_s$, then we write $X_{\mathrm{DTFT}}(f)$ for the DTFT of the samples $x(m/f_s)$; i.e.,

$$X_{\mathrm{DTFT}}(f) := \sum_{n=-\infty}^{\infty} x(n/f_s)e^{-j2\pi fn}.$$

In general, all we can say is that

$$\frac{1}{f_s}X_{\mathrm{DTFT}}(f/f_s) = \sum_{n=-\infty}^{\infty} X(f-nf_s) =: \widetilde{X}(f). \tag{1}$$

However, if the waveform is bandlimited to $f_c < f_s/2$, then

$$\boxed{\frac{1}{f_s}X_{\mathrm{DTFT}}(f/f_s) = X(f), \quad \underline{\underline{|f| \le f_s/2}}.}$$

- ***The Sampling Theorem.*** If $x(t)$ is bandlimited to $f_c$ (remember: $f_c$ is called the **cuttof frequency**) and $f_s > 2f_c$ (remember: $2f_c$ is called the **Nyquist rate**), then **sinc reconstruction formula** or **sinc interpolation formula**

$$x(t) = \sum_{m=-\infty}^{\infty} x(m/f_s)\,\mathrm{sinc}(f_s[t-m/f_s]).$$

If we sample at less than the Nyquist rate, or if the waveform is not even bandlimited, the sinc reconstruction formula yields not $x(t)$ but a distorted or **aliased** version of $x(t)$, namely, the inverse Fourier transform of $\widetilde{X}(f)$. In the frequency domain, aliasing can be intepreted as the overlap of the shifted copies $X(f-nf_s)$ in (1).

- ***The Continuous-Time Domain and the Discrete-Time Domain.*** In the study of the discrete-time processing of *bandlimited* continuous-time signals, it is important to be able to think about a signal in the continuous-time domain and in the discrete-time domain in terms of the signal samples. This relationship, which is given by the boxed equation, is illustrated graphically in Figure 1.

The effect of sampling, that is, to pass from $x(t)$ to the samples $x(n/f_s)$, is carried out as follows. First mark off the points $\pm f_s/2$ so as to contain all the nonzero parts of $X(f)$ as shown at the left in Figure 1. Then multiply all the heights by $f_s$ and divide all the frequencies by $f_s$ as shown at the right in the figure.

To go from the sampled signal back to the continuous-time signal, that is, to use the sinc reconstruction formula, amounts to transforming $X_{\mathrm{DTFT}}(f)$ into $X(f)$ by dividing the heights of the DTFT by $f_s$ and multiplying the frequencies by $f_s$.

- ***The Zero-Order Hold.*** Put $T_s := 1/f_s$. In this reconstruction of $x(t)$ from its samples $x(m/f_s)$, when $mT_s \le t < (m+1)T_s$, we output the constant value $x(m/f_s)$. **In other words, the zero-order hold provides a *staircase approximation* of $x(t)$.** If the zero-order hold approximation is then lowpass filtered by the right ideal filter, $x(t)$ can be exactly recovered. If a nonideal filter is used, very good approximations can be obtained, especially if $f_s \gg f_c$.



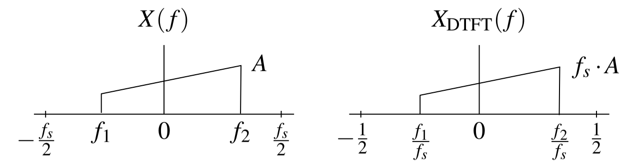


Figure 1. For a bandlimited signal that is sampled at faster than the Nyquist rate, it is easy to pass back and forth between the continuous-time Fourier transform of the signal and the DTFT of its samples.

- ***Bandlimited Waveforms and Systems.*** If a signal and linear time-invariant system are both bandlimited to $f_c$ and $f_s > 2f_c$, then

$$\int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau = \frac{1}{f_s}\sum_{m=-\infty}^{\infty} h(t-m/f_s)x(m/f_s). \tag{2}$$

Since the output of a bandlimited system is bandlimited, there is no loss of information if we restrict attention to the output at the sample times $t = n/f_s$. In other words,

$$\int_{-\infty}^{\infty} h(n/f_s-\tau)x(\tau)\,d\tau = \frac{1}{f_s}\sum_{m=-\infty}^{\infty} h([n-m]/f_s)x(m/f_s)$$

provides enough information to recover (2) for all $t$.

- ***Discrete-Time Convolution.***
($i$) If $x_n$ and $y_n$ are two finite-duration sequences, then their convolution has duration

$$\mathrm{length}(x) + \mathrm{length}(y) - 1.$$

($ii$) When convolving an infinite-duration signal $x_n$ and a finite-duration signal $y_n$, if you only care about the $(x*y)_n$ for finitely many $n$, the problem can be reduced to the convolution of two finite-duration signals.
($iii$) When convolving an infinite-duration signal $x_n$ and a finite-duration signal $y_n$, if you care about $(x*y)_n$ for a large or indeterminate number of $n$, you can do it by convolving $y_n$ with finite-duration blocks from $x_n$ using the **overlap-and-add method**.
($iv$) *How* you choose to compute the convolution of two finite-duration sequences is up to you, but the *fastest* method is to compute the IFFT of the product of FFTs *of zero-padded versions* of the input sequences.

- ***The DTFT and the DFT (or FFT).*** For a finite sequence $y_0,\ldots,y_{M-1}$ with DTFT $Y(f)$, the DFT (or FFT) of $y_0,\ldots,y_{M-1},0,\ldots,0$ that has been zero padded to length $N$ is equal to $Y(k/N)$. In other words, the DFT gives you samples of $Y(f)$. By padding with more and more zeros, you get samples of $Y(f)$ at more closely spaced frequencies.

- ***Window Techniques.*** Multiplying a signal $x_n$ (typically of infinite duration) by a window $w_n$ (typically of finite duration) results in a new signal whose DTFT is the convolution of the DTFT of $x_n$ and the DTFT of $w_n$. The DTFT of $w_n$ is viewed as a not-too-good approximation of an impulse, and so the new signal's spectrum is a smeared or blurred version of that of $x_n$. Windows with a narrow main lobe are good for detecting sinusoids close in frequency and in magnitude. Windows with low side lobes are good for detecting sinusoids that have very different magnitudes; however, such windows have wide main lobes, and so the frequencies you can find cannot be too close.

# Fall 2007 Practice Questions for Exam 1

## Problems

1. Consider a periodic, continuous-time signal $x(t)$ with period $T_0$. Given any sampling period $T_s$, we could obtain samples $x(mT_s)$. Now consider a longer sampling period $T_s' = T_s + L\Delta$, where $\Delta > 0$ and $L$ is any positive integer. Find $\Delta$ such that $x(mT_s') = x(mT_s)$ for all $m$.

2. Let $x(t)$ be a continuous-time, periodic signal with period one and Fourier series coefficients $x_n$. Let $y(t) := x(t/T)$. Find the period of $y(t)$. How are the Fourier series coefficients of $y(t)$ related to the $x_n$?

3. Let $x_n$ have DTFT $X(f)$. Express the DTFTs of $y_n := x_n e^{j2\pi f_0 n}$ and $z_n := x_n \cos(2\pi f_0 n)$ in terms of $X(f)$.

4. Let $x_n$ have DTFT $X(f)$, and let $y_n := x_{n-M}$. Determine whether or not $|Y(f)| = |X(f)|$.

5. If $x$, $y$, and $z$ are sequences with lengths $N_x$, $N_y$, and $N_z$, find the length of $x * y * z$.

6. Let $v$ be the rectangular window of length $M$, and put $w_n := (v * v * v)_n$. Let $N$ denote the length of $w$.
   (a) Express $M$ as a function of $N$.
   (b) Find the main-lobe width.
   (c) Find the (approximate) side-lobe level in dB.

7. Let $x(t)$ have Fourier transform $X(f) = (1+f)/2$ for $|f| \le 1$.
   (a) By hand, sketch $X(f)$ for $|f| \le 4$.
   (b) What is the Nyquist rate?
   (c) By hand, on one graph, sketch $X(f+f_s)$, $X(f)$, and $X(f-f_s)$ when $f_s$ is equal to the Nyquist rate.
   (d) Repeat part (c) if $f_s$ is just a little bit less than the Nyquist rate.

8. Let $x(t) = e^{j2\pi t}$.
   (a) By hand, sketch $X(f)$.
   (b) By hand, sketch $\widetilde{X}(f)$ for $|f| \le f_s/2$ if $f_s = 1.5$.
   (c) If the sinc reconstruction formula is applied to the samples $x(m/f_s)$ with $f_s = 1.5$, what signal will be reconstructed?

9. Show that $\left|\dfrac{\operatorname{sinc}(N\theta/2)}{\operatorname{sinc}(\theta/2)}\right|$ has period 2.



10. Suppose that $x(t) = y(t) + v(t)$, where $y(t)$ is bandlimited to $f_c = 1$ and $v(t)$ is a bandpass signal with $V(f) = 0$ for $|f| < 2$ and $|f| > 3$; i.e., $V(f)$ is nonzero only for $2 < |f| < 3$. We consider $y(t)$ to be the desired signal and $v(t)$ to be unwanted noise. Instead of passing $x(t)$ through a continuous-time lowpass filter to remove $v(t)$ and leave only $y(t)$, you are to design a system that samples $x(t)$ and passes the samples $x(n/f_s)$ through a discrete-time ideal lowpass filter of impulse response $h_n$ and corresponding DTFT $H(f)$ (of period one) such that the output of the discrete-time filter is $y(n/f_s)$, which can then be passed through a D/A to supply $y(t)$. $(i)$ What sampling rate will you use? $(ii)$ Sketch your choice of $H(f)$.