ECE 431

Solutions of Practice Questions for Exam 1

1) Since $x(mT_s') = x(m[T_s + L\Delta]) = x(mT_s + mL\Delta)$, it is clear that is $L\Delta$ is a multiple k of the period T_0, then $x(mT_s') = x(mT_s + mkT_0)$ $= x(mT_s)$. So, all we need is $L\Delta = kT_0$. For example, $L=1$ and $\Delta = T_0$ will suffice.

2) If x has period one, then $y(t) := x(t/T)$ has period T. To prove this, write

$$y(t+T) = x\left(\frac{t+T}{T}\right) = x(t/T + 1) = x(t/T) = y(t),$$

The Fourier series coefficients of $y(t)$ are the same as those of $x(t)$; i.e., $y_n = x_n$. There are two ways to see this. First, since

$$x(t) = \sum_n x_n e^{+j2\pi nt},$$

$$y(t) = x(t/T) = \sum_n x_n e^{j2\pi nt/T}$$

Since x_n is the coeff. of $e^{j2\pi nt/T}$, $y_n = x_n$. The second way to see that $y_n = x_n$ is to write

$$y_n = \frac{1}{T}\int_{-T/2}^{T/2} y(t)\, e^{-j2\pi nt/T} = \frac{1}{T}\int_{-T/2}^{T/2} x(t/T) e^{-j2\pi nt/T}\, dt$$

Now let $\theta = t/T$, $d\theta = dt/T$ to get

$$y_n = \int_{-1/2}^{1/2} x(\theta)\, e^{-j2\pi n\theta}\, d\theta = x_n.$$

3) $Y(f) = \sum_n y_n e^{-j2\pi f n} = \sum_n \left(x_n e^{j2\pi f_0 n} \right) e^{-j2\pi f n}$

$= \sum_n x_n e^{-j2\pi (f-f_0) n} = X(f-f_0)$

Since $z_n = x_n \cos(2\pi f_0 n) = x_n \left[e^{j2\pi f_0 n} + e^{-j2\pi f_0 n} \right]/2$,
$Z(f) = \frac{1}{2}\left[X(f-f_0) + X(f+f_0) \right]$.

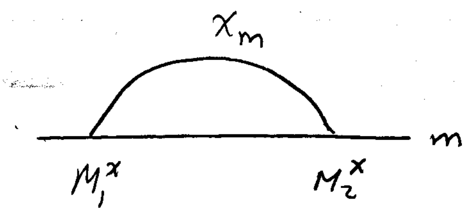
4) $Y(f) = \sum_{n=-\infty}^{\infty} x_{n-M} e^{-j2\pi f n} = \sum_{k=-\infty}^{\infty} x_k e^{-j2\pi f [k+M]}$

$k = n - M$

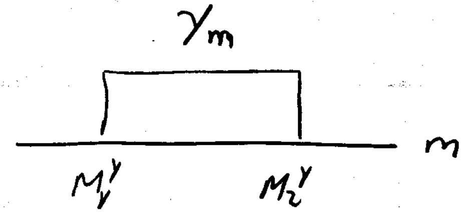
$= e^{-j2\pi f M} X(f)$.

So, $|Y(f)| = |X(f)|$.

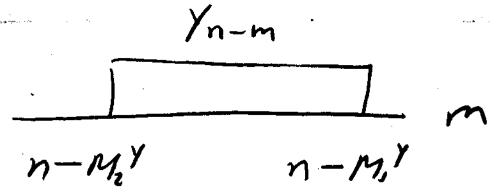
5)



$\Rightarrow N_x = M_2^x - M_1^x + 1$



$N_y = M_2^y - M_1^y + 1$



From the pictures, $(x*y)_n$ lives from

$n - M_1^y = M_1^x$ to

$n - M_2^y = M_2^x$

or $M_1^x + M_1^y \le n \le M_2^x + M_2^y$

$\Rightarrow \text{length}(x*y) = (M_2^x + M_2^y) - (M_1^x + M_1^y) + 1$

$= (M_2^x - M_1^x + 1) + (M_2^y - M_1^y + 1) - 1$

$= \text{length}(x) + \text{length}(y) - 1$

So, $\text{length}(x * y) = N_x + N_y - 1$.

So, $\text{length}((x*y)*z)) = [N_x + N_y - 1] + N_z - 1$

$$= N_x + N_y + N_z - 2$$

6) $w = v * v * v \Rightarrow N = 3M - 2$ or $M = \dfrac{N+2}{3}$

(a)

$$W(f) = V(f)^3$$

$$= \left(\frac{\sin\left(\pi \frac{N+2}{3} f\right)}{\sin(\pi f)} \right)^3 e^{-j\pi f (N-1)}$$

(b) The first zero of $W(f)$ occurs when

$\dfrac{N+2}{3} f = 1$ or $f = \dfrac{3}{N+2}$

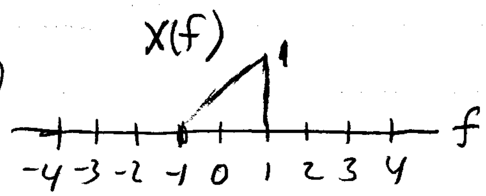
So, the main-lobe width $= \dfrac{6}{N+2}$

(c) Since the second zero is at $2 \times \frac{3}{N+2}$, the side-lobe level will be approximately

$$20 \log_{10} \frac{\left| W\left(1.5 \frac{3}{N+2}\right) \right|}{|W(0)|} \approx 20 \log_{10} \left| \frac{(-1)/\left(\frac{4.5\pi}{N+2}\right)}{\frac{N+2}{3}} \right|^3$$

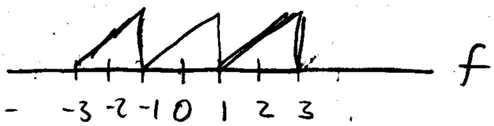
$$= 60 \log_{10} \left(\frac{2}{3\pi} \right) \approx -40.39 \text{ dB}$$

7) (a)

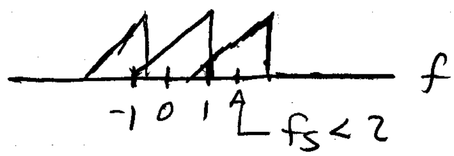


(b) Nyquist rate $= 2$.

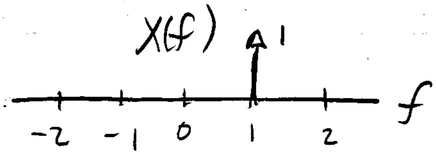
7) c

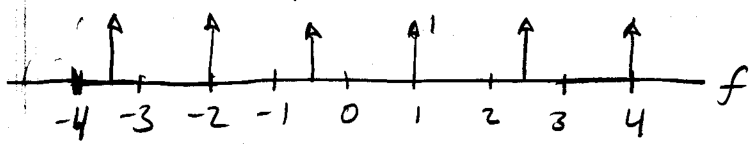


d



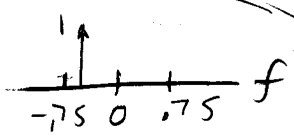
8) a





$\leftarrow \tilde{X}(f)$

b



$\leftarrow$ $\tilde{X}(f)$ for $|f| \le \frac{f_s}{2} = .75$

c $e^{-j2\pi(.75)t}$ will be reconstructed.

9) First write

$$\left|\frac{\operatorname{sinc}(N\theta/2)}{\operatorname{sinc}(\theta/2)}\right| = \left|\frac{\sin(\pi N\theta/2)}{\pi N\theta/2} \cdot \frac{\pi\theta/2}{\sin(\pi\theta/2)}\right|$$

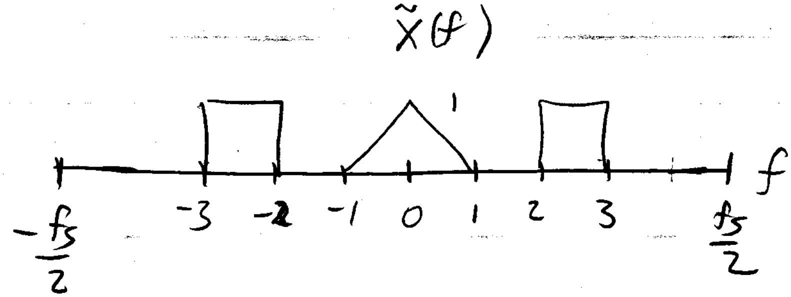
$$= \frac{1}{N}\left|\frac{\sin(\pi N\theta/2)}{\sin(\pi\theta/2)}\right|$$

Now, $\sin(\pi N[\theta+2]/2) = \sin(\pi N\theta/2 + \pi N) = \sin(\pi N\theta/2)\cos(\pi N)$

and $\sin(\pi[\theta+2]/2) = \sin(\pi\theta/2 + \pi) = \sin(\pi\theta/2)\cos(\pi)$.

Finally $|\cos(\pi N)| = |(-1)^N| = 1 = |(-1)| = |\cos(\pi)|$.

10) Since the description of $x(t)$ implies it is bandlimited to $f_{xc} = 3$, we must choose $f_s \geq 2f_{xc} = 6$. Then for $|f| \leq f_s/2$,



Since $\tilde{X}(f) = \frac{1}{f_s} X_{DTFT}(f/f_s)$, $X_{DTFT}(f) = f_s \tilde{X}(f_s \cdot f)$.

But a simple sketch expresses this much better: