This last example illustrates the general case. If

$$R_1 := \inf\left\{ r \geq 0 : \sum_{n=-\infty}^{\infty} |x_n| r^{-n} < \infty \right\} \quad \text{and} \quad R_2 := \sup\left\{ r \geq 0 : \sum_{n=-\infty}^{\infty} |x_n| r^{-n} < \infty \right\},$$

then

$$\{z : R_1 < |z| < R_2\} \subseteq \text{ROC}(x).$$

An important observation is that the ROC contains all circles of the form $\{z : |z| = r\}$ for $R_1 < r < R_2$. The ROC may or may not contain some z with $|z| = R_1$ or $|z| = R_2$. In Example 5.2, $R_1 = 0$, $R_2 = |a|$, and $z = 0$ is in the ROC. In Example 5.1, $R_1 = |a|$, $R_2 = \infty$, and $\lim_{z\to\infty} X(z) = x_0$; i.e., $z = \infty$ is in the ROC.

5.1.1. Importance of the ROC

Let $x_n^+ := a^n$ for $n \geq 0$ and $x_n^+ = 0$ for $n < 0$. Let $x_n^- := -a^n$ for $n < 0$ and $x_n^- = 0$ for $n \geq 0$. Then by Example 5.1, the z transform of x_n^+ is $1/(1 - az^{-1})$. By the calculations in Example 5.3 (or Problem 1), the z transform of x_n^- is also $1/(1 - az^{-1})$. What's going on?

The "catch" is that we have ignored the region of convergence. In the case of x_n^+, the ROC is $|z| > |a|$, while in the case of x_n^-, the ROC is $|z| < |a|$. The ROC is part of the z transform of a signal. It is not enough just to give the formula; you must also say for what values of z the formula holds — you must also give the ROC.

To say that two z transforms are the same means that they have the same ROC and that their formulas are equal to each other for all z in the ROC. So, two transforms are different if they have different ROCs, even if their formulas are the same. Two transforms are different if their ROCs are the same but their formulas are not equal for some z in their common ROC.

5.1.2. The Inverse z Transform

As mentioned above, the ROC always contains circles of the form $\{z : |z| = r\}$ for $R_1 < r < R_2$. For such r, consider evaluating the z transform at points z of the form $z = re^{j2\pi f}$, where $|f| \leq 1/2$. We get

$$X(z)\big|_{z=re^{j2\pi f}} = \sum_{n=-\infty}^{\infty} x_n \big(re^{j2\pi f}\big)^{-n} = \sum_{n=-\infty}^{\infty} x_n r^{-n} e^{-j2\pi f n}.$$

We recognize this as the DTFT of the sequence $x_n r^{-n}$. It follows by the inverse DTFT that

$$x_n r^{-n} = \int_{-1/2}^{1/2} X\big(re^{j2\pi f}\big) e^{j2\pi f n}\, df,$$

or

$$x_n = r^n \int_{-1/2}^{1/2} X\big(re^{j2\pi f}\big) e^{j2\pi f n}\, df.$$

Problems

1. If $x_n = -a^n$ for $n < 0$ and $x_n = 0$ for $n \geq 0$, show that its z transform is

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a|.$$