CHAPTER 7
Random Processes: Basic Concepts

A **random process** or **stochastic process** is any family of random variables. However, this terminology is usually applied only to infinite families of random variables.

The simplest example of a random process is an infinite sequence of i.i.d. random variables, say $X_n \sim \text{Bernoulli}(p)$ for $n = 1, 2, \ldots$, or $X_n \sim N(0,1)$ for $n = 0, \pm 1, \pm 2 \ldots$. A family of random variables indexed by an infinite subset of the integers is called a **discrete-time random processes**.

In this chapter, we focus on **continuous-time random processes**, X_t, where $-\infty < t < \infty$. Most of what follows can be carried out for discrete-time processes X_n, $-\infty < n < \infty$, in a similar way.

Suppose that for each $-\infty < t < \infty$, X_t is a random variable. Remember, this means that X_t is a function defined on the sample space Ω. In other words, for each $\omega \in \Omega$, $X_t(\omega)$ is a real number. However, it is often more convenient to regard ω as fixed and to let t vary. From this perspective, for each choice of ω, we obtain a waveform $X_t(\omega)$ as we let t vary. Hence, we often think of X_t as a random waveform.

Chapter Summary

Section 7.1 introduces the mean function, the correlation function, and the covariance function of a random process. Section 7.2 introduces the concept of a stationary process and of a wide-sense stationary process. The correlation function and the power spectral density of wide-sense stationary processes are introduced and their properties are derived. Section 7.3 is the heart of the chapter, and covers the analysis of wide-sense stationary processes through linear, time-invariant systems. These results are then applied in Sections 7.4 and 7.5 to derive the matched filter and the Wiener filter. Section 7.6 contains a discussion of the expected time-average power in a wide-sense stationary random process. The Wiener–Khinchin Theorem is derived and used to provide an alternative expression for the power spectral density.

7.1. Mean, Correlation, and Covariance

If X_t is a random process, its **mean function** is

$$m_X(t) := \mathsf{E}[X_t].$$

Its **correlation function** is

$$R_X(t_1,t_2) := \mathsf{E}[X_{t_1}X_{t_2}].$$

Note that

$$R_X(t_1,t_2) = \mathsf{E}[X_{t_1}X_{t_2}] = \mathsf{E}[X_{t_2}X_{t_1}] = R_X(t_2,t_1).$$

In other words, the correlation function is a **symmetric function** of t_1 and t_2. The correlation function has other important properties as well. For example, $R_X(t,t) = \mathsf{E}[X_t^2] \ge 0$, and for any t_1 and t_2,

$$\big|R_X(t_1,t_2)\big| \le \sqrt{\mathsf{E}[X_{t_1}^2]\,\mathsf{E}[X_{t_2}^2]}. \tag{7.1}$$

This is an easy consequence of the **Cauchy–Schwarz inequality** (see Problem 1), which says that for any random variables U and V,

$$\mathsf{E}[UV]^2 \le \mathsf{E}[U^2]\,\mathsf{E}[V^2].$$

Taking $U = X_{t_1}$ and $V = X_{t_2}$ yields the desired result.

The **covariance function** is

$$C_X(t_1,t_2) := \mathsf{E}\big[(X_{t_1} - \mathsf{E}[X_{t_1}])(X_{t_2} - \mathsf{E}[X_{t_2}])\big],$$

An easy calculation shows that

$$C_X(t_1,t_2) = R_X(t_1,t_2) - m_X(t_1)m_X(t_2).$$

Note that the covariance function is also symmetric; i.e., $C_X(t_1,t_2) = C_X(t_2,t_1)$.

Since we usually assume that our processes are zero mean; i.e., $m_X(t) \equiv 0$, we focus on the correlation function and its properties.

7.2. Wide-Sense Stationary Processes

A random process is **stationary** or **strictly stationary** if for any finite collection of times $t_1, \ldots, t_n$, all joint probabilities involving $X_{t_1}, \ldots, X_{t_n}$ are the same as those involving $X_{t_1+\Delta t}, \ldots, X_{t_n+\Delta t}$ for any time shift Δt. This implies, for example, that $\wp(X_{t_1} \in B) = \wp(X_{t_2} \in B)$ for any set B. In other words, $\wp(X_t \in B)$ does not depend on t. Similarly, stationarity implies that for any function $g(x)$, $\mathsf{E}[g(X_{t_1})] = \mathsf{E}[g(X_{t_2})]$, and $\mathsf{E}[g(X_t)]$ does not depend on t. Stationarity also implies that for any function $g(x_1,x_2)$, we have

$$\mathsf{E}[g(X_{t_1},X_{t_2})] = \mathsf{E}[g(X_{t_1+\Delta t},X_{t_2+\Delta t})]$$

for every time shift Δt. Since Δt is arbitrary, let $\Delta t = -t_2$. Then

$$\mathsf{E}[g(X_{t_1},X_{t_2})] = \mathsf{E}[g(X_{t_1-t_2},X_0)].$$

It follows that $\mathsf{E}[g(X_{t_1},X_{t_2})]$ depends on t_1 and t_2 only through the time difference $t_1 - t_2$.

Asking that all the joint probabilities involving any $X_{t_1}, \ldots, X_{t_n}$ be the same as those involving $X_{t_1+\Delta t}, \ldots, X_{t_n+\Delta t}$ for every time shift Δt is a very strong requirement. In practice, e.g., analyzing receiver noise in a communication system, it is often enough to require only that $\mathsf{E}[X_t]$ not depend on t and that the correlation $R_X(t_1,t_2) = \mathsf{E}[X_{t_1}X_{t_2}]$ depend on t_1 and t_2 only through the time difference, $t_1 - t_2$. This is a much weaker requirement than stationarity for several reasons. First, we are only concerned with one or two variables at a time rather than arbitrary finite collections. Second, we are not concerned with probabilities, only expectations. Third, we are only concerned with $\mathsf{E}[X_t]$ and $\mathsf{E}[X_{t_1}X_{t_2}]$ rather than $\mathsf{E}[g(X_t)]$ and $\mathsf{E}[g(X_{t_1},X_{t_2})]$ for arbitrary functions g.

We say that a process is **wide-sense stationary** (**WSS**) if its mean function $m_X(t) = \mathsf{E}[X_t]$ does not depend on t *and* if its correlation function $R_X(t_1, t_2) = \mathsf{E}[X_{t_1} X_{t_2}]$ depends on t_1 and t_2 only through the time difference $t_1 - t_2$.

***Example* 7.1.** In modeling a communication system, the carrier signal at the receiver is modeled by $X_t = \cos(2\pi f t + \Theta)$, where $\Theta \sim \text{uniform}[-\pi, \pi]$. The reason for the random phase is that the receiver does not know the exact time when the transmitter was turned on. Show that X_t is WSS.

Solution. We must show that the mean $\mathsf{E}[X_t]$ does not depend on t, and we must show that $\mathsf{E}[X_{t_1} X_{t_2}]$ is a function of $t_1 - t_2$. For the mean, write

$$\begin{aligned} \mathsf{E}[X_t] &= \mathsf{E}[\cos(2\pi f t + \Theta)] \\ &= \int_{-\infty}^{\infty} \cos(2\pi f t + \theta)\, f_\Theta(\theta)\, d\theta \\ &= \int_{-\pi}^{\pi} \cos(2\pi f t + \theta) \frac{d\theta}{2\pi}. \end{aligned}$$

Be careful to observe that this last integral is with respect to θ, *not* t. Hence, this integral evaluates to zero, which does not depend on t.

For the correlation, write

$$\begin{aligned} R_X(t_1, t_2) &= \mathsf{E}[X_{t_1} X_{t_2}] \\ &= \mathsf{E}\big[\cos(2\pi f t_1 + \Theta)\cos(2\pi f t_2 + \Theta)\big] \\ &= \frac{1}{2}\mathsf{E}\big[\cos(2\pi f[t_1 + t_2] + 2\Theta) \\ &\qquad + \cos(2\pi f[t_1 - t_2])\big]. \end{aligned}$$

The first cosine has expected value zero just as the mean did. The second cosine is nonrandom, and equal to its expected value. Thus, $R_X(t_1, t_2) = \cos(2\pi f[t_1 - t_2])/2$ as required.

Once we know a process is WSS, we write $R_X(t_1 - t_2)$ instead of $R_X(t_1, t_2)$. We can also write

$$R_X(\tau) = \mathsf{E}[X_{t+\tau} X_t], \quad \text{for any choice of } t.$$

In particular, note that

$$R_X(0) = \mathsf{E}[X_t^2] \geq 0, \quad \text{for all } t.$$

At time t, the instantaneous power in a WSS process is X_t^2. The expected instantaneous power is then

$$P_X := \mathsf{E}[X_t^2] = R_X(0),$$

which does not depend on t, since the process is WSS.

It is now convenient to introduce the Fourier transform of $R_X(\tau)$,

$$S_X(f) := \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau}\, d\tau.$$

We call $S_X(f)$ the **power spectral density** of the process. Observe that by the Fourier inversion formula,

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau}\, df.$$

In particular, taking $\tau = 0$, shows that

$$\int_{-\infty}^{\infty} S_X(f)\, df = R_X(0) = P_X.$$

To explain the terminology, "power spectral density," recall that *probability densities* are nonnegative functions that are integrated to obtain probabilities. Similarly, *power* spectral *densities* are nonnegative functions* that are integrated to obtain powers. The adjective "spectral" refers to the fact that $S_X(f)$ is a function of frequency.

***Example* 7.2.** Let X_t be a WSS random process with power spectral density $S_X(f) = e^{-f^2/2}$. Find the power in the process.

Solution. The power is

$$\begin{aligned} P_X &= \int_{-\infty}^{\infty} S_X(f)\, df \\ &= \int_{-\infty}^{\infty} e^{-f^2/2}\, df \\ &= \sqrt{2\pi} \int_{-\infty}^{\infty} \frac{e^{-f^2/2}}{\sqrt{2\pi}}\, df \\ &= \sqrt{2\pi}. \end{aligned}$$

***Example* 7.3.** Let X_t be a WSS process with correlation function $R_X(\tau) = 5\sin(\tau)/\tau$. Find the power in the process.

Solution. The power is

$$P_X = R_X(0) = \lim_{\tau \to 0} R_X(\tau) = \lim_{\tau \to 0} 5\frac{\sin(\tau)}{\tau} = 5.$$

***Example* 7.4.** Let X_t be a WSS random process with ideal lowpass power spectral density $S_X(f) = I_{[-W,W]}(f)$ shown in Figure 7.1. Find its correlation function $R_X(\tau)$.

Solution. We must find the inverse Fourier transform of $S_X(f)$. Write

$$\begin{aligned} R_X(\tau) &= \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau}\, df \\ &= \int_{-W}^{W} e^{j2\pi f \tau}\, df \end{aligned}$$

*The nonnegativity of power spectral densities is derived in Example 7.7.

$$
\begin{aligned}
&= \left.\frac{e^{j2\pi f\tau}}{j2\pi\tau}\right|_{f=-W}^{f=W} \\
&= \frac{e^{j2\pi W\tau} - e^{-j2\pi W\tau}}{j2\pi\tau} \\
&= 2W\frac{e^{j2\pi W\tau} - e^{-j2\pi W\tau}}{2j(2\pi W\tau)} \\
&= 2W\frac{\sin(2\pi W\tau)}{2\pi W\tau}.
\end{aligned}
$$

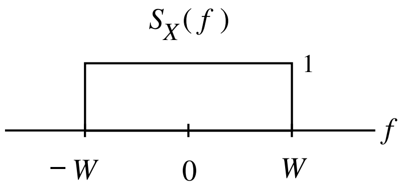

***Figure* 7.1.** Power spectral density of bandlimited noise in Example 7.4.

If the bandwidth W of $S_X(f)$ in the preceding example is allowed to go to infinity, then all frequencies will be present in equal amounts. A WSS process whose power spectral density is constant for all f is call **white noise**. The value of the constant is usually denoted by $\mathcal{N}_0/2$. Since the inverse transform of a constant function is a delta function, the correlation function of white noise with $S_X(f) = \mathcal{N}_0/2$ is $R_X(\tau) = \frac{\mathcal{N}_0}{2}\delta(\tau)$.

Properties of Correlation Functions and Power Spectral Densities

The correlation function $R_X(\tau)$ is completely characterized by the following three properties.[1]

(i) $R_X(-\tau) = R_X(\tau)$; i.e., R_X is an even function of τ.
(ii) $|R_X(\tau)| \le R_X(0)$. In other words, the maximum value of $|R_X(\tau)|$ occurs at $\tau = 0$. In particular, taking $\tau = 0$ reaffirms that $R_X(0) \ge 0$.
(iii) The power spectral density, $S_X(f)$, is a real, even, nonnegative function of f.

Property (i) is a restatement of the fact any correlation function, as a function of two variables, is symmetric; for a WSS process this means that $R_X(t_1 - t_2) = R_X(t_2 - t_1)$. Taking $t_1 = \tau$ and $t_2 = 0$ yields the result.

Property (ii) is a restatement of (7.1) for a WSS process. However, we can also derive it directly by again using the Cauchy–Schwarz inequality. Write

$$
\begin{aligned}
\big|R_X(\tau)\big| &= \big|\mathsf{E}[X_{t+\tau}X_t]\big| \\
&\le \sqrt{\mathsf{E}[X_{t+\tau}^2]\,\mathsf{E}[X_t^2]} \\
&= \sqrt{R_X(0)\,R_X(0)} \\
&= R_X(0).
\end{aligned}
$$

Property (iii): To see that $S_X(f)$ is real and even, write

$$
\begin{aligned}
S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}\,dt \\
&= \int_{-\infty}^{\infty} R_X(\tau)\cos(2\pi f\tau)\,d\tau \\
&\quad - j\int_{-\infty}^{\infty} R_X(\tau)\sin(2\pi f\tau)\,d\tau.
\end{aligned}
$$

Since correlation functions are real and even, and since the sine is odd, the second integrand is odd, and therefore integrates to zero. Hence, we can always write

$$
S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\cos(2\pi f\tau)\,d\tau.
$$

Thus, $S_X(f)$ is real. Furthermore, since the cosine is even, $S_X(f)$ is an even function of f as well.

The fact that $S_X(f)$ is nonnegative is derived later in Example 7.7.

Remark. Property (iii) actually implies Properties (i) and (ii). (Problem 8.) Hence, a real-valued function of τ is a correlation function if and only if its Fourier transform is real, even, and nonnegative. However, if one is trying to show that a function of τ is *not* a correlation function, it is easier to check if either (i) or (ii) fails. Of course, if (i) and (ii) both hold, it is then necessary to go ahead and find the power spectral density and see if it is nonnegative.

***Example* 7.5.** Determine whether or not $R(\tau) := \tau e^{-|\tau|}$ is a valid correlation function.

Solution. Property (i) requires that $R(\tau)$ be even. However, $R(\tau)$ is not even (in fact, it is odd). Hence, it cannot be a correlation function.

***Example* 7.6.** Determine whether or not $R(\tau) := 1/(1+\tau^2)$ is a valid correlation function.

Solution. We must check the three properties that characterize correlation functions. It is obvious that R is even and that $\big|R(\tau)\big| \le R(0) = 1$. The Fourier transform of $R(\tau)$ is $S(f) = \pi\exp(-2\pi|f|)$, as can be verified by applying the inversion formula to $S(f)$.† Since $S(f)$ is nonnegative, we see that $R(\tau)$ satisfies all three properties, and is therefore a valid correlation function.

7.3. WSS Processes through Linear Time-Invariant Systems

In this section, we consider passing a WSS random process through a **linear time-invariant** (**LTI**) **system**. Our goal is to find the correlation and power spectral density of

†There is a short table of transform pairs in the Problems section.

the output. Before doing so, it is convenient to introduce the following concepts.

Let X_t and Y_t be random processes. Their **cross-correlation function** is

$$R_{XY}(t_1,t_2) \;:=\; \mathsf{E}[X_{t_1}Y_{t_2}].$$

To distinguish between the terms cross-correlation function and correlation function, the latter is sometimes referred to as the **auto-correlation function**. The **cross-covariance function** is

$$\begin{aligned} C_{XY}(t_1,t_2) &= \mathsf{E}[\{X_{t_1}-m_X(t_1)\}\{Y_{t_2}-m_Y(t_2)\}] \\ &= R_{XY}(t_1,t_2) - m_X(t_1)\,m_Y(t_2). \end{aligned}$$

A pair of processes X_t and Y_t is called **jointly wide-sense stationary (J-WSS)** if X_t is WSS, Y_t is WSS *and* if the cross-correlation $R_{XY}(t_1,t_2) = \mathsf{E}[X_{t_1}Y_{t_2}]$ depends on t_1 and t_2 only through the difference $t_1 - t_2$. If this is the case, we write $R_{XY}(t_1-t_2)$ instead of $R_{XY}(t_1,t_2)$.

Recall that an LTI system is expressed by the convolution

$$Y_t \;=\; \int_{-\infty}^{\infty} h(t-\tau)X_\tau\,d\tau,$$

where h is the impulse response of the system. We now show that if the input X_t is WSS, then the output Y_t is WSS and X_t and Y_t are J-WSS.

To begin, we show that $\mathsf{E}[Y_t]$ does not depend on t. To do this, write

$$\mathsf{E}[Y_t] \;=\; \mathsf{E}\left[\int_{-\infty}^{\infty} h(t-\tau)X_\tau\,d\tau\right] \;=\; \int_{-\infty}^{\infty} \mathsf{E}[h(t-\tau)X_\tau]\,d\tau.$$

In computing the expectation inside the integral, observe that X_τ is random, but $h(t-\tau)$ is not. Hence, $h(t-\tau)$ is a constant and can be pulled out of the expectation; i.e., $\mathsf{E}[h(t-\tau)X_\tau] = h(t-\tau)\mathsf{E}[X_\tau]$. Now, since X_t is WSS, $\mathsf{E}[X_\tau]$ does not depend on τ and is equal some constant, say m. Hence,

$$\mathsf{E}[Y_t] \;=\; \int_{-\infty}^{\infty} h(t-\tau)\,m\,d\tau.$$

On this integral, make the change of variable $\theta = t-\tau$, $d\theta = -d\tau$. Then

$$\mathsf{E}[Y_t] \;=\; m\int_{-\infty}^{\infty} h(\theta)\,d\theta,$$

which does not depend on t.

Logically, the next step would be to show that $\mathsf{E}[Y_{t_1}Y_{t_2}]$ depends on t_1 and t_2 only through the difference t_1-t_2. However, it is more efficient if we first obtain a formula for the cross-correlation, $\mathsf{E}[X_{t_1}Y_{t_2}]$, and show that it depends only on t_1-t_2. Write

$$\begin{aligned} \mathsf{E}[X_{t_1}Y_{t_2}] &= \mathsf{E}\left[X_{t_1}\int_{-\infty}^{\infty} h(t_2-\tau)X_\tau\,d\tau\right] \\ &= \int_{-\infty}^{\infty} h(t_2-\tau)\mathsf{E}[X_{t_1}X_\tau]\,d\tau \\ &= \int_{-\infty}^{\infty} h(t_2-\tau)R_X(t_1-\tau)\,d\tau, \end{aligned}$$

since X_t is WSS. Now make the change of variable $\theta = t_1-\tau$, $d\theta = -d\tau$ to obtain

$$\mathsf{E}[X_{t_1}Y_{t_2}] \;=\; \int_{-\infty}^{\infty} h(-[\{t_1-t_2\}-\theta])R_X(\theta)\,d\theta,$$

which depends only on t_1-t_2 as required. We can now write

$$R_{XY}(\tau) \;=\; \int_{-\infty}^{\infty} h(-[\tau-\theta])R_X(\theta)\,d\theta. \qquad (7.2)$$

In other words, R_{XY} is the convolution of $h(-\theta)$ and $R_X(\theta)$. This suggests that we define the **cross power spectral density** by

$$S_{XY}(f) \;:=\; \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j2\pi f\tau}\,d\tau.$$

Since we are considering real-valued random variables here, we must assume $h(\theta)$ is real valued. Letting $H(f)$ denote the Fourier transform of $h(\theta)$, it follows that the Fourier transform of $h(-\theta)$ is $H(f)^*$, where the superscript $*$ denotes the complex conjugate. It now follows that

$$S_{XY}(f) \;=\; H(f)^*S_X(f). \qquad (7.3)$$

We now calculate the auto-correlation of Y_t and show that it depends only on t_1-t_2. Write

$$\begin{aligned} \mathsf{E}[Y_{t_1}Y_{t_2}] &= \mathsf{E}\left[\left(\int_{-\infty}^{\infty} h(t_1-\tau)X_\tau d\tau\right)Y_{t_2}\right] \\ &= \int_{-\infty}^{\infty} h(t_1-\tau)\mathsf{E}[X_\tau Y_{t_2}]\,d\tau \\ &= \int_{-\infty}^{\infty} h(t_1-\tau)R_{XY}(\tau-t_2)\,d\tau. \end{aligned}$$

Now make the change of variable $\theta = \tau - t_2$, $d\theta = d\tau$ to obtain

$$\mathsf{E}[Y_{t_1}Y_{t_2}] \;=\; \int_{-\infty}^{\infty} h(t_1-t_2-\theta)R_{XY}(\theta)\,d\theta.$$

Thus,

$$R_Y(\tau) \;=\; \int_{-\infty}^{\infty} h(\tau-\theta)R_{XY}(\theta)\,d\theta.$$

In other words, R_Y is the convolution of h and R_{XY}. Taking Fourier transforms, we obtain $S_Y(f) = H(f)S_{XY}(f) = H(f)H(f)^*S_X(f)$, and

$$S_Y(f) \;=\; \left|H(f)\right|^2 S_X(f). \qquad (7.4)$$

In other words, the power spectral density of the output of an LTI system is the product of the magnitude squared of the transfer function and the power spectral density of the input.

***Example* 7.7.** We can use the above formula to show that power spectral densities are nonnegative. Suppose X_t is a WSS process with $S_X(f) < 0$ on some interval

$[W_1, W_2]$, where $0 < W_1 \le W_2$. Since S_X is even, it is also negative on $[-W_2, -W_1]$ as shown in Figure 7.2. Let $H(f) := I_{[-W_2,-W_1]}(f) + I_{[W_1,W_2]}(f)$ as shown in Figure 7.2. Note that since $H(f)$ takes only the values zero and one, $|H(f)|^2 = H(f)$. Let $h(t)$ denote the inverse Fourier transform of H. Since H is real and even, h is also real and even. Then $Y_t := \int_{-\infty}^{\infty} h(t-\tau)X_\tau\, d\tau$ is a real WSS random process, and

$$S_Y(f) \;=\; |H(f)|^2 S_X(f) \;=\; H(f)S_X(f) \;\le\; 0,$$

with $S_Y(f) < 0$ for $W_1 < |f| < W_2$. Hence, $\int_{-\infty}^{\infty} S_Y(f)\,df < 0$. It then follows that

$$0 \;\le\; \mathsf{E}[Y_t^2] \;=\; R_Y(0) \;=\; \int_{-\infty}^{\infty} S_Y(f)\,df \;<\; 0,$$

which is a contradiction. Thus, $S_X(f) \ge 0$.

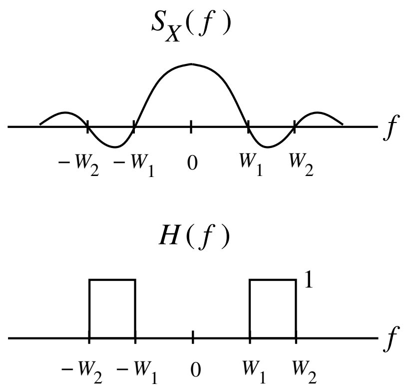

***Figure* 7.2.** Setup of Example 7.7 to show that a power spectral density must be nonnegative.

***Example* 7.8.** A certain communication receiver employs a bandpass filter to reduce white noise generated in the amplifier. Suppose that the white noise X_t has power spectral density $\mathcal{N}_0/2$ and that the filter transfer function $H(f)$ is given in Figure 7.2. Find the output power from the filter.

Solution. The output power is obtained by integrating the power spectral density of the filter output. Denoting the filter output by Y_t,

$$P_Y \;=\; \int_{-\infty}^{\infty} S_Y(f)\,df \;=\; \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\,df.$$

Since $|H(f)|^2 S_X(f) = \mathcal{N}_0/2$ for $W_1 \le |f| \le W_2$, and is zero otherwise, $P_Y = 2(\mathcal{N}_0/2)(W_2 - W_1) = \mathcal{N}_0(W_2 - W_1)$. In other words, the output power is $\mathcal{N}_0$ times the bandwidth of the filter.

7.4. The Matched Filter

Consider an air-traffic control system which sends out a known radar pulse. If there are no objects in range of the radar, the system returns only a noise waveform X_t, which we model as a zero-mean, WSS random process with power spectral density $S_X(f)$. If there is an object in range, the system returns the reflected radar pulse, say $v(t)$, plus the noise X_t. We wish to design a system that decides whether received waveform is noise only, X_t, or signal plus noise, $v(t) + X_t$. As an aid to achieving this goal, we propose to take the received waveform and pass it through an LTI system with impulse response $h(t)$. If the received signal is in fact $v(t) + X_t$, then the output of the linear system is

$$\int_{-\infty}^{\infty} h(t-\tau)[v(\tau) + X_\tau]\,d\tau \;=\; v_0(t) + Y_t,$$

where

$$v_0(t) \;:=\; \int_{-\infty}^{\infty} h(t-\tau)v(\tau)\,d\tau$$

is the output signal, and

$$Y_t \;:=\; \int_{-\infty}^{\infty} h(t-\tau)X_\tau\,d\tau$$

is the output noise process. We will now try to find the impulse response h that maximizes the output **signal-to-noise ratio (SNR)**,

$$\text{SNR} \;:=\; \frac{v_0(t_0)^2}{\mathsf{E}[Y_{t_0}^2]},$$

where $v_0(t_0)^2$ is the instantaneous output signal power at time t_0, and $\mathsf{E}[Y_{t_0}^2]$ is the expected instantaneous output noise power at time t_0. Note that since $\mathsf{E}[Y_{t_0}^2] = R_Y(0) = P_Y$, we can also write

$$\text{SNR} \;=\; \frac{v_0(t_0)^2}{P_Y}.$$

Our approach will be to obtain an upper bound on the numerator of the form $v_0(t_0)^2 \le P_Y \cdot B$, where B does not depend on the impulse response h. It will then follow that

$$\text{SNR} \;=\; \frac{v_0(t_0)^2}{P_Y} \;\le\; \frac{P_Y \cdot B}{P_Y} \;=\; B.$$

We then show how to choose the impulse response so that in fact $v_0(t_0)^2 = P_Y \cdot B$. For this choice of impulse response, we then have $\text{SNR} = B$, the maximum possible value.

We begin by analyzing the denominator in the SNR. Observe that

$$P_Y \;=\; \int_{-\infty}^{\infty} S_Y(f)\,df \;=\; \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\,df.$$

To analyze the numerator, write

$$v_0(t_0) \;=\; \int_{-\infty}^{\infty} V_0(f)e^{j2\pi f t_0}\,df \;=\; \int_{-\infty}^{\infty} H(f)V(f)e^{j2\pi f t_0}\,df,$$

where $V_0(f)$ is the Fourier transform of $v_0(t)$, and $V(f)$ is the Fourier transform of $v(t)$. Next, write

$$\begin{aligned} v_0(t_0) &= \int_{-\infty}^{\infty} H(f)\sqrt{S_X(f)} \cdot \frac{V(f)e^{j2\pi f t_0}}{\sqrt{S_X(f)}}\,df \\ &= \int_{-\infty}^{\infty} H(f)\sqrt{S_X(f)} \cdot \left[\frac{V(f)^* e^{-j2\pi f t_0}}{\sqrt{S_X(f)}}\right]^* df, \end{aligned}$$

where the asterisk denotes complex conjugation. Applying the Cauchy–Schwarz inequality (see Problem 1 and the Remark following it), we obtain the upper bound,

$$|v_0(t)|^2 \le \underbrace{\int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\,df}_{=P_Y} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{|V(f)|^2}{S_X(f)}\,df}_{=:B}.$$

Thus,

$$\mathrm{SNR} = \frac{|v_0(t_0)|^2}{P_Y} \le \frac{P_Y \cdot B}{P_Y} = B.$$

Now, the Cauchy–Schwarz inequality holds with equality if and only if $H(f)\sqrt{S_X(f)}$ is a multiple of $V(f)^* e^{-j2\pi f t_0}/\sqrt{S_X(f)}$. Thus, the upper bound on the SNR will be achieved if we take $H(f)$ to solve

$$H(f)\sqrt{S_X(f)} = \alpha \frac{V(f)^* e^{-j2\pi f t_0}}{\sqrt{S_X(f)}}$$

for any complex multiplier α; i.e., we should take

$$H(f) = \alpha \frac{V(f)^* e^{-j2\pi f t_0}}{S_X(f)}.$$

Thus, the optimal filter is "matched" to the known signal.

Consider the special case in which X_t is white noise with power spectral density $S_X(f) = \mathcal{N}_0/2$. Taking $\alpha = \mathcal{N}_0/2$ as well, we have $H(f) = V(f)^* e^{-j2\pi f t_0}$, which inverse transforms to $h(t) = v(t_0 - t)$, assuming $v(t)$ is real. Thus, the matched filter has an impulse response which is a time-reversed and translated copy of the known signal $v(t)$. An example of $v(t)$ and the corresponding $h(t)$ is shown in Figure 7.3. As the figure illustrates, if $v(t)$ is a finite-duration, causal waveform, as any radar "pulse" would be, then the sampling time t_0 can always be chosen so that $h(t)$ corresponds to a causal system.

7.5. The Wiener Filter

Omitted.

7.6. Expected Time-Average Power and the Wiener–Khinchin Theorem

Omitted.

7.7. Notes

Notes §7.2: Wide-Sense Stationary Processes

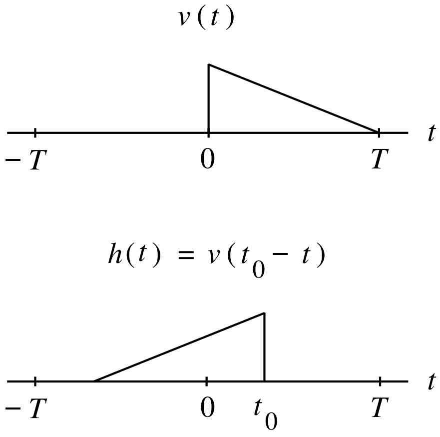

***Figure* 7.3.** Known signal $v(t)$ and corresponding matched filter impulse response $h(t)$ in the case of white noise.

***Note* 1.** We mean that if R_X is the correlation function of a WSS process, then R_X it must satisfy Properties (*i*)–(*iii*). Furthermore, it can be proved [30, pp. 94–96] that if R is a real-valued function satisfying Properties (*i*)–(*iii*), then there is a WSS process having R as its correlation function.

7.8. Problems

Recall that if

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi f t}\,dt,$$

then the inversion formula is

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi f t}\,df.$$

With these formulas in mind, the following table of Fourier transform pairs may be useful.

$h(t)$	$H(f)$
$e^{-\lambda t}u(t)$	$\dfrac{1}{\lambda + j2\pi f}$
$e^{-\lambda\|t\|}$	$\dfrac{2\lambda}{\lambda^2 + (2\pi f)^2}$
$\dfrac{\lambda}{\lambda^2 + t^2}$	$\pi\, e^{-2\pi\lambda\|f\|}$
$e^{-(t/\sigma)^2/2}$	$\sqrt{2\pi}\,\sigma\, e^{-\sigma^2(2\pi f)^2/2}$

Problems §7.1: Mean, Correlation, and Covariance

1. The Cauchy–Schwarz inequality says that for any random variables U and V,

$$\mathsf{E}[UV]^2 \le \mathsf{E}[U^2]\,\mathsf{E}[V^2]. \tag{7.5}$$

Derive this formula as follows. For any constant λ, we can always write

$$\begin{aligned} 0 &\leq \mathsf{E}[|U-\lambda V|^2] \qquad (7.6)\\ &= \mathsf{E}[U^2] - 2\lambda\mathsf{E}[UV] + \lambda^2\mathsf{E}[V^2]. \end{aligned}$$

Now set $\lambda = \mathsf{E}[UV]/\mathsf{E}[V^2]$ and rearrange the inequality. Note that since equality holds in (7.6) if and only if $U = \lambda V$, equality holds in (7.5) if and only if U is a multiple of V.

Remark. The same technique can be used to show that for complex-valued waveforms,

$$\left|\int_{-\infty}^{\infty} g(\theta)\, h(\theta)^*\, d\theta\right|^2 \leq \int_{-\infty}^{\infty} |g(\theta)|^2\, d\theta \cdot \int_{-\infty}^{\infty} |h(\theta)|^2\, d\theta,$$

where the asterisk denotes complex conjugation, and for any complex number z, $|z|^2 = z \cdot z^*$.

2. Let X_t for $t > 0$ be a random process with zero mean and correlation function $R_X(t_1, t_2) = \min(t_1, t_2)$. If X_t is Gaussian for each t, write down the density of X_t.

Problems §7.2: Wide-Sense Stationary Processes

3. Find the correlation function corresponding to each of the following power spectral densities. (a) $\delta(f)$. (b) $\delta(f-f_0) + \delta(f+f_0)$. (c) $e^{-f^2/2}$. (d) $e^{-|f|}$.

4. Let X_t be a WSS random process with power spectral density $S_X(f) = I_{[-W,W]}(f)$. Find $\mathsf{E}[X_t^2]$.

5. Explain why each of the following frequency functions cannot be a power spectral density. (a) $e^{-f}u(f)$, where u is the unit step function. (b) $e^{-f^2}\cos(f)$. (c) $(1-f^2)/(1+f^4)$. (d) $1/(1+jf^2)$.

6. For each of the following functions, determine whether or not it is a valid correlation function.
(a) $\sin(\tau)$. (b) $\cos(\tau)$. (c) $e^{-\tau^2/2}$. (d) $e^{-|\tau|}$. (e) $\tau^2 e^{-|\tau|}$. (f) $I_{[-T,T]}(\tau)$.

7. Let $R_0(\tau)$ be a correlation function, and put $R(\tau) := R_0(\tau)\cos(2\pi f_0 \tau)$ for some $f_0 > 0$. Determine whether or not $R(\tau)$ is a valid correlation function.

8. Let $S(f)$ be a real-valued, even, nonnegative function, and put

$$R(\tau) := \int_{-\infty}^{\infty} S(f) e^{j2\pi f\tau}\, d\tau.$$

Show that R satisfies properties (i) and (ii) that characterize a correlation function.

9. Let $R_0(\tau)$ be a real-valued, even function, but not necessarily a correlation function. Let $R(\tau)$ denote the convolution of R_0 with itself, i.e.,

$$R(\tau) := \int_{-\infty}^{\infty} R_0(\theta)\, R_0(\tau-\theta)\, d\theta.$$

(a) Show that $R(\tau)$ is a valid correlation function. *Hint:* You will need the remark made in Problem 1.

(b) Now suppose that $R_0(\tau) = I_{[-T,T]}(\tau)$. In this case, what is $R(\tau)$, and what is its Fourier transform?

Problems §7.3: WSS Processes through LTI Systems

10. A WSS input signal X_t with correlation function $R_X(\tau) = e^{-\tau^2/2}$ is passed through an LTI system with transfer function $H(f) = e^{-(2\pi f)^2/2}$. Denote the system output by Y_t. Find (a) the cross power spectral density, $S_{XY}(f)$; (b) the cross-correlation, $R_{XY}(\tau)$; (c) $\mathsf{E}[X_{t_1}Y_{t_2}]$; (d) the output power spectral density, $S_Y(f)$; (e) the output auto-correlation, $R_Y(\tau)$; (f) the output power P_Y.

11. White noise with power spectral density $S_X(f) = \mathcal{N}_0/2$ is passed though a lowpass RC filter with impulse response $h(t) = \frac{1}{RC}e^{-t/(RC)}u(t)$. Find (a) the cross power spectral density, $S_{XY}(f)$; (b) the cross-correlation, $R_{XY}(\tau)$; (c) $\mathsf{E}[X_{t_1}Y_{t_2}]$; (d) the output power spectral density, $S_Y(f)$; (e) the output auto-correlation, $R_Y(\tau)$; (f) the output power P_Y.

12. White noise with power spectral density $S_X(f) = \mathcal{N}_0/2$ is passed though a filter with impulse response $h(t) = I_{[-T/2,T/2]}(t)$. Find and sketch the correlation function of the filter output.

13. Let X_t be a WSS random process, and put $Y_t := \int_{t-3}^{t} X_\tau\, d\tau$. Determine whether or not Y_t is WSS.

14. Let X_t be a WSS random process. Put $Y_t := \int_{-\infty}^{\infty} h(t-\tau)X_\tau\, d\tau$, and $Z_t := \int_{-\infty}^{\infty} g(t-\theta)X_\theta\, d\theta$. Determine whether or not Y_t and Z_t are J-WSS.

15. Let X_t be a zero-mean WSS random process with power spectral density $S_X(f) = 2/[1+(2\pi f)^2]$. Put $Y_t := X_t - X_{t-1}$.

(a) Show that X_t and Y_t are J-WSS.

(b) Find the power spectral density $S_Y(f)$.

(c) Find the power in Y_t.

16. *(Extension to Complex Random Processes).* If X_t is a complex-valued random process, then its auto-correlation function is defined by $R_X(t_1,t_2) := \mathsf{E}[X_{t_1}X_{t_2}^*]$. Similarly, if Y_t is another complex-valued random process, their cross-correlation is defined by $R_{XY}(t_1,t_2) := \mathsf{E}[X_{t_1}Y_{t_2}^*]$. The concepts of WSS, J-WSS, the power spectral density, and the cross power spectral density are defined as in the real case. Now suppose that X_t is a complex WSS process and that $Y_t = \int_{-\infty}^{\infty} h(t-\tau)X_\tau\, d\tau$, where the impulse response h is now possibly complex valued.

(a) Show that $R_X(-\tau) = R_X(\tau)^*$.

(b) Show that $S_X(f)$ must be real valued. *Hint:* What does part (a) say about the real and imaginary parts of $R_X(\tau)$?

(c) Show that

$$\mathsf{E}[X_{t_1} Y_{t_2}^*] = \int_{-\infty}^{\infty} h(-[\{t_1 - t_2\} - \theta])^* R_X(\theta)\, d\theta.$$

(d) Even though the above result is a little different from (7.2), show that (7.3) and (7.4) still hold for complex random processes.

Problems §7.4: The Matched Filter

17. Determine the matched filter if the known radar pulse is $v(t) = \sin(t) I_{[0,\pi]}(t)$, and X_t is white noise with power spectral density $S_X(f) = \mathcal{N}_0/2$. For what values of t_0 is the optimal system causal?

18. Determine the matched filter if $v(t) = e^{-(t/\sqrt{2})^2/2}$, and $S_X(f) = e^{-(2\pi f)^2/2}$.

Problems §7.5: The Wiener Filter

19. Suppose V_t and X_t are J-WSS. Let $U_t := V_t + X_t$. Show that U_t and V_t are J-WSS.

20. Suppose V_t and X_t are each zero mean and WSS. Also assume that $\mathsf{E}[V_t X_\tau] = 0$ for all t and τ. Express the Wiener filter $H(f)$ in terms of $S_V(f)$ and $S_X(f)$.