ECE 729
Decoders and Continuous Channels Subject to an Input Power Constraint

1. Decoders for General Channels

Let $B_1,\dots,B_N$ be subsets of some space $\boldsymbol{Y}$. Then put

$$\begin{aligned} F_1 &:= B_1 \\ F_i &:= B_i \cap B_{i-1}^c \cap \cdots \cap B_1^c, \quad i = 2,\dots,N. \end{aligned}$$

Also put

$$F := \left(\bigcup_{i=1}^N F_i\right)^c = F_1^c \cap \cdots \cap F_N^c,$$

and note that $F \cap F_i = \varnothing$ for $i = 1,\dots,N$. In addition,

$$F_1 \cup \cdots \cup F_N \cup F = \boldsymbol{Y},$$

and so $F_1,\dots,F_N$ and F constitute a **partition** of $\boldsymbol{Y}$.

Consider the formula

$$\varphi(\boldsymbol{y}) := \sum_{i=1}^N i I_{F_i}(\boldsymbol{y}) + N I_F(\boldsymbol{y}).$$

Then for $i = 1,\dots,N-1$, $\varphi(\boldsymbol{y}) = i \Leftrightarrow \boldsymbol{y} \in F_i$, and thus

$$\varphi(\boldsymbol{y}) \neq i \Leftrightarrow \boldsymbol{y} \in F_i^c. \tag{1}$$

Since $\varphi(\boldsymbol{y}) = N \Leftrightarrow \boldsymbol{y} \in F_N \cup F$, we have

$$\varphi(\boldsymbol{y}) \neq N \Leftrightarrow \boldsymbol{y} \in F_N^c \cap F^c \subset F_N^c. \tag{2}$$

Hence, (1) holds *even when* $i = N$. To conclude, observe that $F_1^c = B_1^c$, and for $2 \le i \le N$,

$$F_i^c = B_i^c \cup \left(\bigcup_{j<i} B_j\right).$$

We can now write *for all* $i = 1,\dots,N$,

$$B_i^c \subset F_i^c \subset B_i^c \cup \left(\bigcup_{j\neq i} B_j\right). \tag{3}$$

It follows that

$$\boxed{\{\boldsymbol{y} : \varphi(\boldsymbol{y}) \neq i\} \subset B_i^c \cup \left(\bigcup_{j\neq i} B_j\right), \quad \underline{\underline{i = 1,\dots,N.}}} \tag{4}$$

We also have

$$\boxed{\{\boldsymbol{y} : \varphi(\boldsymbol{y}) \neq i\} = F_i^c \supset B_i^c, \quad \underline{\underline{i = 1,\dots,N-1.}}} \tag{5}$$

2. Continuous Channels Subject to an Input Power Constraint

Let A_n be as in the notes and put

$$B_i := \{\boldsymbol{y} : (\boldsymbol{x}_i, \boldsymbol{y}) \in A_n \text{ and } \|\boldsymbol{x}_i\|^2 \le nP\},$$

and define the F_i and φ as above. Then

$$B_i^c = \{\boldsymbol{y} : (\boldsymbol{x}_i, \boldsymbol{y}) \notin A_n\} \cup \{\boldsymbol{y} : \|\boldsymbol{x}_i\|^2 > nP\}, \tag{6}$$

and

$$B_j \subset \{\boldsymbol{y} : (\boldsymbol{x}_j, \boldsymbol{y}) \in A_n\}. \tag{7}$$

In addition, note from (6) that

$$B_i^c \supset \{\boldsymbol{y} : \|\boldsymbol{x}_i\|^2 > nP\}. \tag{8}$$

We first use (4) to write

$$\begin{aligned} \int_{\{\boldsymbol{y}:\varphi(\boldsymbol{y})\neq i\}} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\,d\boldsymbol{y} &\le \int_{B_i^c \cup \left(\bigcup_{j\neq i} B_j\right)} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\,d\boldsymbol{y} \\ &\le \int_{B_i^c} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\,d\boldsymbol{y} + \sum_{j\neq i} \int_{B_j} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\,d\boldsymbol{y}. \end{aligned}$$

Next, by (6),

$$\begin{aligned} \int_{B_i^c} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\,d\boldsymbol{y} &\le \underbrace{\int_{\{\boldsymbol{y}:\|\boldsymbol{x}_i\|^2 > nP\}} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\,d\boldsymbol{y}}_{=:\,\eta(\boldsymbol{x}_i)} \\ &\quad + \underbrace{\int_{\{\boldsymbol{y}:(\boldsymbol{x}_i,\boldsymbol{y})\in A_n\}} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\,d\boldsymbol{y}}_{=:\,\alpha(\boldsymbol{x}_i)}, \end{aligned}$$

and by (7),

$$\int_{B_j} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\,d\boldsymbol{y} \le \int_{\{\boldsymbol{y}:(\boldsymbol{x}_j,\boldsymbol{y})\in A_n\}} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\,d\boldsymbol{y} =: \theta(\boldsymbol{x}_i, \boldsymbol{x}_j).$$

We can now write

$$\boxed{\int_{\{\boldsymbol{y}:\varphi(\boldsymbol{y})\neq i\}} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\,d\boldsymbol{y} \le \eta(\boldsymbol{x}_i) + \alpha(\boldsymbol{x}_i) + \sum_{j\neq i} \theta(\boldsymbol{x}_i, \boldsymbol{x}_j),}$$

where η, α, and θ are as in the notes.

Let $0 < \lambda < 1/2$ and $\Delta R > 0$ be given. Suppose

$$\frac{\log N}{n} > R - \Delta R/2$$

and that from $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$, we have at least $\lfloor N/2 \rfloor$ codewords that satisfy

$$\int_{\{\boldsymbol{y}:\varphi(\boldsymbol{y})\neq i\}} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\, d\boldsymbol{y} < 2\lambda < 1. \tag{9}$$

If $i \neq N$, we claim that $\|\boldsymbol{x}_i\|^2 \leq nP$. Otherwise, the set on the right in (8) is equal to $\mathbb{R}^n$, and then by (5),

$$\int_{\{\boldsymbol{y}:\varphi(\boldsymbol{y})\neq i\}} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\, d\boldsymbol{y} \geq \int_{\mathbb{R}^n} w^n(\boldsymbol{y}|\boldsymbol{x}_i)\, d\boldsymbol{y} = 1,$$

which contradicts (9). Hence, there are at least $\lfloor N/2 \rfloor - 1$ codewords that satisfy (9) *and* $\|\boldsymbol{x}_i\|^2 \leq nP$. This does not affect the rate of the code since

$$\begin{aligned}
\frac{\log\big(\lfloor N/2 \rfloor - 1\big)}{n} &\geq \frac{\log\big(\{N/2-1\}-1\big)}{n} = \frac{\log(N/2-2)}{n} \\
&\geq \frac{\log(N/2-N/4)}{n} = \frac{\log(N/4)}{n} \\
&= \frac{\log N}{n} - \frac{\log 4}{n} \\
&> (R - \Delta R/2) - \frac{\log 4}{n} \geq R - \Delta R
\end{aligned}$$

for large n.