Recall that for variable-length codes, the goal was to minimize the expected code-length, \( E[\ell(C_n(X_1, \ldots, X_n))] \). For fixed-length codes we take a different point of view. Suppose \( X < \infty \). Then there are \( |X|^n = \exp(n \log |X|) \) different possible realizations of the random source sequence \((X_1, \ldots, X_n)\). Our goal here is to find a small subset of \( X^n \) that has small cardinality and large probability. Somewhat more precisely, we want to find a set \( A_n \subset X^n \) with
\[
|A_n| < \exp(n \log |X|) \quad \text{and} \quad P((X_1, \ldots, X_n) \in A_n) \approx 1.
\]
Our results will be asymptotic in that we obtain both conditions together only when the blocklength \( n \) is sufficiently large.

**Definition 4.2.** A real number \( R \geq 0 \) is an **achievable rate** for a source \( X_1, X_2, \ldots \) if \( \forall \Delta R > 0, \forall \lambda > 0, \exists n_0 \) such that \( \forall n \geq n_0, \exists A_n \subset X^n \) with
\[
\frac{\log |A_n|}{n} \leq R + \Delta R \quad \text{and} \quad P((X_1, \ldots, X_n) \notin A_n) < \lambda.
\]
Note that the first condition is equivalent to \( |A_n| \leq \exp(n[R + \Delta R]) \).

**Theorem 4.3** (Shannon’s Source Coding Theorem). For a DMS with pmf \( p \), we have the following.

(i) **(Forward result):** Every rate \( R \geq H(p) \) is achievable

(ii) **(Converse result):** Every rate \( 0 \leq R < H(p) \) is not achievable.

**Proof.** First, observe that if \( R \) is achievable, any \( R' \geq R \) is also achievable. Hence, it suffices to prove \( H(p) \) is an achievable rate. For \( \Delta R > 0 \), consider the set
\[
A_n := \left\{ x \in X^n : \left| \frac{1}{n} \sum_{k=1}^{n} \log \frac{1}{p(x_k)} - H(p) \right| \leq \Delta R \right\}.
\]
Let \( X = (X_1, \ldots, X_n) \). Observe that
\[
x \in A_n \iff -\Delta R \leq \frac{1}{n} \sum_{k=1}^{n} \log \frac{1}{p(x_k)} - H(p) \leq \Delta R
\]
\[
\iff H(p) - \Delta R \leq \frac{1}{n} \log \frac{1}{p(x_1) \cdots p(x_n)} \leq H(p) + \Delta R
\]
\[
\iff \exp(-n[H(p) - \Delta R]) \geq P(X = x) \geq \exp(-n[H(p) + \Delta R]).
\]
Next,
\[
1 \geq P(X \in A_n) = \sum_{x \in A_n} P(X = x) \\
\leq \sum_{x \in A_n} \exp(-n[H(p) + \Delta R]) \\
= |A_n| \exp(-n[H(p) + \Delta R]).
\]

So, \(\exp(n[H(p) + \Delta R]) \geq |A_n|\), or \(\frac{\log |A_n|}{n} \leq H(p) + \Delta R\). This satisfies the first property needed to show that \(H(p)\) is an achievable rate.

To demonstrate the second property needed to show that \(H(p)\) is an achievable rate, we simply note that
\[
P(X \notin A_n) = P\left(|\frac{1}{n} \sum_{k=1}^{n} \log \frac{1}{p(X_k)} - H(p)| > \Delta R\right) \rightarrow 0
\]

by the **weak law of large numbers (WLLN)** [1, p. 576]. Thus, \(H(p)\) is an achievable rate.

**References**


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\(a\) The WLLN says that if \(Y_1, Y_2, \ldots\) are i.i.d. with finite mean \(E[Y_1]\), then for every \(\varepsilon > 0\),
\[
P\left(\left|\frac{1}{n} \sum_{k=1}^{n} Y_k - E[Y_1]\right| > \varepsilon\right) \rightarrow 0 \text{ as } n \rightarrow \infty.
\]