ECE 729, Lec. 1
Exam 1
20 March 2001

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may use a calculator.

Some Formulas

- The log inequality:
$$\log\theta \le (\log e)(\theta - 1).$$
- The binary entropy function is defined by
$$h(\theta) := -[\theta\log\theta + (1-\theta)\log(1-\theta)],$$
and its derivative is
$$h'(\theta) = -(\log e)\ln\Big(\frac{\theta}{1-\theta}\Big).$$
- Average mutual information:
$$I(X\wedge Y) := \sum_x\sum_y P_{XY}(x,y)\log\frac{P_{XY}(x,y)}{P_X(x)P_Y(y)}.$$
- The capacity of the binary symmetric channel (BSC) is $1-h(\varepsilon)$ bits per channel use.

1. [15 pts.] Let $\mathsf{X} := \{1,2,3,4,5,6\}$. The probabilities of points in X are given by

x	$\mathsf{P}(X = x)$
1	0.25
2	0.20
3	0.15
4	0.15
5	0.13
6	0.12

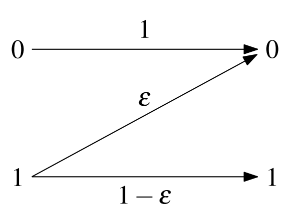
Construct a binary Huffman code, and compute its expected length in bits.

2. Which of the following are true/false? You do NOT need to justify your answer.
   (a) [5 pts.] $I(X \wedge Z|Y) \geq I(Z \wedge Y|X) - I(Z \wedge Y) + I(X \wedge Z)$.
   (b) [5 pts.] $I(X \wedge Y) \geq I(X \wedge Y|Z)$.
   (c) [5 pts.] $H(XYZ) - H(XY) \leq H(XZ) - H(X)$.

3. Consider a binary, memoryless source with $\mathsf{P}(X_n = 1) = 9/10$ and $\mathsf{P}(X_n = 0) = 1/10$.
   (a) [5 pts.] Find the entropy of the source.
   (b) [10 pts.] It is desired to find a block source code of rate $1/3$ whose probability of error is less than 0.20. Can this be done? **Justify your answer.**
   (c) [10 pts.] It is desired to find a block source code of rate $2/3$ whose probability of error is less than 0.20. Can this be done? **Justify your answer.**

4. [20 pts.] Your company has a large contract to provide a channel code for a BSC with crossover probability $\varepsilon = 1/8$. System constraints require an $n = 255$-bit codeword. The information to be transmitted will be blocks of i.i.d. bits, U_i, with $\mathsf{P}(U_i = 1) = \mathsf{P}(U_i = 0) = 1/2$. The blocks $(U_1, \ldots, U_k)$ will be combined with $n - k$ parity bits to create an n-bit channel codeword. What is the largest value of k that you would consider? Justify your answer.

5. [25 pts.] Consider the discrete memoryless Z channel:



Letting $p = P_X(1)$, find p to maximize $I(X \wedge Y)$.