ECE 729, Lec. 1
Final Exam
14 May 2001

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring one sheet of 8.5× 11 paper on which you have prepared formulas.

You may use a calculator.

Some Formulas

- The *entropy rate* of a stationary source $\{X_n\}$ is

$$\mathscr{H} \;=\; \lim_{n\to\infty} \tfrac{1}{n} H(X_1 \cdots X_n) \;=\; \lim_{n\to\infty} H(X_n | X_{n-1} \cdots X_1).$$

- The capacity of the ideal bandlimited Gaussian channel is $W \log_2\left(1 + \frac{P}{\mathscr{N} W}\right)$ bits per second.

1. [20 pts.] It is desired to transmit 20 Mbits/s over an ideal bandlimited Gaussian channel of bandwidth 1.5 MHz and $\mathscr{N} = 10^{-9}$ Watts/Hz. Determine the minimum power that must be supplied by the transmitter.

2. [20 pts.] Let $\{X_n\}$ be a discrete, stationary source with entropy rate $\mathscr{H}$. Show that for any fixed $k = 1, 2, 3, \ldots,$

$$\lim_{n\to\infty} \tfrac{1}{n} H(X_n \cdots X_1 | X_0 X_{-1} \cdots X_{-k}) \;=\; \mathscr{H}.$$

3. [20 pts.] Let X and Y be finite sets, and let W be a transition probability from X into Y. If p is a probability mass function on X, let $\mathscr{R}(p)$ denote the interval

$$\mathscr{R}(p) \;:=\; [0, I(p \times W)].$$

Let

$$\mathscr{R} \;:=\; \bigcup_p \mathscr{R}(p).$$

Show that $\mathscr{R}$ is a closed set. **Justify your answer.**

4. [20 pts.] Recall that the rate-distortion function is defined by

$$R(D) \;:=\; \inf\{R \geq 0 : (R, D) \text{ is achievable } \}.$$

Use the fact that the set of achievable rate-distortion pairs (R, D) is convex to prove that $R(D)$ is a convex function of D.

5. [20 pts.] Your boss asks you to evaluate the following opportunity. Your company has been offered a contract to build a compression system for an i.i.d. Gaussian source of zero mean and variance $\sigma^2 = 16$ subject to squared-error distortion. The system is required to achieve a compression rate of $R = 3$ bits per source sample and an average distortion of $D = 1/2$. The rate-distortion function is

$$R(D) \;=\; \begin{cases} \frac{1}{2}\log_2\left(\frac{\sigma^2}{D}\right), & 0 \leq D \leq \sigma^2, \\ 0, & D > \sigma^2. \end{cases}$$

The contract allows for a large amount of money for any necessary research and development. Your boss would like to know if it is theoretically possible to design and build a system that meets the contract requirements. What would you tell your boss? **Justify your answer.**