ECE 729, Lec. 1
Exam 1
Wednesday, 24 March 2004
5–6:15 pm in 2345 EH

100 Points   5 Questions

Justify your answers!   **Be precise!**

Closed Book   **Closed Notes**

You may use a calculator.

Some Formulas

- The log inequality:
$$\log\theta \leq (\log e)(\theta - 1).$$
- The binary entropy function is defined by
$$h(\theta) := -[\theta\log\theta + (1-\theta)\log(1-\theta)],$$
and its derivative is
$$h'(\theta) = -(\log e)\ln\Big(\frac{\theta}{1-\theta}\Big).$$
- Average mutual information:
$$I(X \wedge Y) := \sum_x \sum_y P_{XY}(x,y)\log\frac{P_{XY}(x,y)}{P_X(x)P_Y(y)}.$$
- The capacity of the binary symmetric channel (BSC) is $1 - h(\varepsilon)$ bits per channel use.

1. [15 pts.] Let $\mathsf{X} := \{1,2,3,4,5,6\}$. The probabilities of points in X are given by

x	$\mathsf{P}(X = x)$
1	0.30
2	0.25
3	0.20
4	0.10
5	0.08
6	0.07

   Construct a ternary Huffman code, **and** compute its expected length in ternary digits.

2. The source alphabet $\mathsf{X} := \{a,b,c\}$ is to be encoded using the following two variable-length codes:

x	code 1
a	0
b	01
c	11

x	code 2
a	0
b	10
c	11

   For each code, answer the following questions:

   (a) [5 pts.] Is this a prefix code (yes/no)?

   (b) [5 pts.] Is this code uniquely decodable (yes/no)?

   (c) [10 pts.] It is possible to uniquely decode the *infinite* sequence of a 0 followed by all 1s forever, $011111\cdots$ (yes/no)?

3. A ternary DMS produces 512 symbols/second with $\mathsf{P}(U_n = 0) = 1/6$, $\mathsf{P}(U_n = 1) = 1/3$, and $\mathsf{P}(U_n = 2) = 1/2$. These symbols are compressed with a source code. The compressed data is communicated over a BSC that operates 1600 times/second. Assume the BSC crossover probability is $\varepsilon = 1/10$.

   (a) [10 pts.] Find the entropy of the source (in bits/source symbol).

   (b) [5 pts.] Find the capacity of the BSC (in bits/channel use).

   (c) [10 pts.] Determine whether or not it is possible to send the source information reliably over the channel with arbitrarily small probability of error. **Justify your answer.**

4. Let X,S,Y, and T have joint pmf of the form

$$P_{XSYT}(x,s,y,t) = r(x,s)W(y|x)V(t|s).$$

   Determine whether or not the following are true or false. Circle your answer <u>and</u>: if the statement is true, derive it; if the statement is false, briefly explain why.

   (a) [10 pts.] $H(YT|XS) = H(Y|X) + H(T|S)$.

   (b) [10 pts.] $I(XS \wedge YT) \leq I(X \wedge Y) + I(S \wedge T)$.

5. [20 pts.] Let X and Y be the binary-valued input and output of a DMC defined as follows. Let X, N_1, and N_2 be independent $\{0,1\}$-valued random variables, and put $Y := X \oplus N_1 \oplus N_2$, where $\oplus$ denotes mod-2 addition (exclusive or):

$$0 \oplus 0 = 0, \quad 1 \oplus 0 = 1, \quad 0 \oplus 1 = 1, \quad \text{and} \quad 1 \oplus 1 = 0.$$

If $\mathsf{P}(N_1 = 1) = \mathsf{P}(N_2 = 1) = \varepsilon$, and if $W(y|x) := \mathsf{P}(Y = y|X = x)$, find the capacity of the DMC W.