ECE 729, Lec. 1
Exam 1
Friday, 24 Mar. 2006
1:20-2:10 pm
3444 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

1. [33 pts.] You are stranded on a desert island without a calculator or tables of logarithms. You need to bound $H(X)$, where X is a discrete random variable with

$$\mathsf{P}(X=1)=0.5, \quad \mathsf{P}(X=2)=0.3, \quad \mathsf{P}(X=3)=0.1, \quad \text{and} \quad \mathsf{P}(X=4)=0.1.$$

Use your knowledge of variable-length source coding to show that $H(X) \le 1.7$ bits. **Justify your answer.**

2. [33 pts.] Show that convolution can only increase differential entropy in the sense that if p and q are densities, then $H(p*q) \ge H(q)$, where

$$(p*q)(y) := \int_{-\infty}^{\infty} p(y-z)q(z)\,dz \quad \text{and} \quad H(q) := \int_{-\infty}^{\infty} q(z)\log\frac{1}{q(z)}\,dz.$$

Hint: Consider independent random variables X and Z where X has density p and Z has density q.

3. [34 pts.] Let W_1 and W_2 be two discrete memoryless channels with capacities

$$C_1 := \sup_q I(X_1 \wedge Y_1) \quad \text{and} \quad C_2 := \sup_q I(X_2 \wedge Y_2),$$

where $P_{X_1Y_1}(x,y) := q(x)W_1(y|x)$ and $P_{X_2Y_2}(x,y) := q(x)W_2(y|x)$. If we define the new DMC

$$W(y_1,y_2|x_1,x_2) := W_1(y_1|x_1)W_2(y_2|x_2),$$

then W can be considered as modeling **parallel** DMCs that are independent. Show that the capacity of W is C_1+C_2; i.e., show that

$$\sup_p I(X_1X_2 \wedge Y_1Y_2) = C_1 + C_2,$$

where

$$P_{X_1X_2Y_1Y_2}(x_1,x_2,y_1,y_2) = p(x_1,x_2)W_1(y_1|x_1)W_2(y_2|x_2).$$

Hints: (i) Show that $I(X_1X_2 \wedge Y_1Y_2) \le I(X_1 \wedge Y_1) + I(X_2 \wedge Y_2)$.
(ii) You may assume that q_1 achieves the supremum defining C_1 and that q_2 achieves the supremum defining C_2.