ECE 729, Lec. 1
Final Exam
Friday, 12 May 2006
7:45–9:45 am
3418 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Open Notes**

- Exponential density with parameter μ: $f(\gamma) = \mu e^{-\mu\gamma}$, $\gamma \geq 0$.
- The binary entropy function: $h(\theta) := -[\theta \log \theta + (1-\theta)\log(1-\theta)]$.

1. ***Background:*** Consider a discrete-time, additive white Gaussian noise channel subject to block fading. This means that when a codeword $(x_1, \ldots, x_n) \in \mathrm{I\!R}^n$ is transmitted, the kth received channel output symbol is $Y_k = \sqrt{G}x_k + Z_k$, where the Z_k are i.i.d. zero-mean Gaussians with common variance σ^2. If the receiver knows the value of G, then the codeword constraint

$$\frac{1}{n}\sum_{k=1}^{n} x_k^2 \leq A \quad \text{is equivalent to} \quad \frac{1}{n}\sum_{k=1}^{n} (\sqrt{G}x_k)^2 \leq GA,$$

and so the capacity is $\frac{1}{2}\log(1 + GA/\sigma^2)$. If we put

$$C(\gamma) := \tfrac{1}{2}\log(1+\gamma),$$

then the capacity is $C(GA/\sigma^2)$. However, since the fading coefficient $\sqrt{G}$ is random, the signal-to-noise ratio (SNR) $\Gamma := GA/\sigma^2$ is also random. If the SNR falls below a threshold γ, we say that an **outage** occurs. The parametric curve

$$\Big(\mathsf{P}(\Gamma \leq \gamma), C(\gamma)\Big), \quad \gamma \geq 0,$$

expresses the capacity $c = C(\gamma)$ as a function of the **outage probability** $p = \mathsf{P}(\Gamma \leq \gamma)$.

Question: Find a formula to express the capacity c as a function of the outage probability p if Γ has an exponential density with parameter μ (Rayleigh fading). Your formula should involve only p and μ.

2. Consider the d-dimensional, discrete-time Gaussian channel defined as follows. The response of the channel to an input symbol $\boldsymbol{x} \in \mathbb{R}^d$ is

$$\boldsymbol{Y} = \boldsymbol{x} + \boldsymbol{Z},$$

where $\boldsymbol{Z}$ is a d-dimensional, zero-mean Gaussian random vector with i.i.d. components such that $\mathsf{E}[\|\boldsymbol{Z}\|^2] = \sigma^2$. In other words, $\boldsymbol{Z}$ has covariance matrix $(\sigma^2/d)I$, where I is the d-dimensional identity matrix. The channel is subject to the following cost constraint. Every codeword $\underline{\boldsymbol{x}} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in (\mathbb{R}^d)^n$ must satisfy

$$\frac{1}{n}\sum_{k=1}^{n} \|\boldsymbol{x}_k\|^2 \leq A.$$

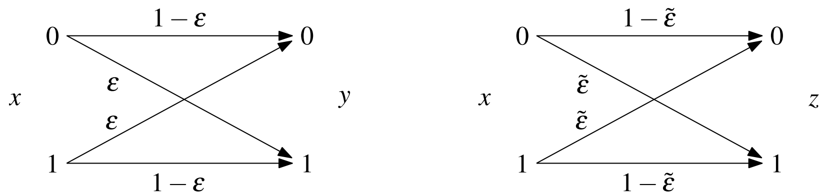
The capacity of this channel is

$$C(A) = \max_{X:\mathsf{E}[\|X\|^2]\leq A} I(\boldsymbol{X} \wedge \boldsymbol{Y}),$$

where it is understood that $\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{Z}$ with $\boldsymbol{X}$ and $\boldsymbol{Z}$ independent. Find $C(A)$ in terms of A, σ^2, and d.

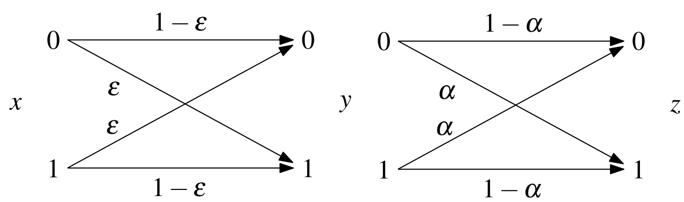
Remarks: (i) When the base of the logarithm is two, the capacity has units of bits per channel use. However, since each channel use transmits d real numbers, $C(A)/d$ is the capacity in bits per real number transmitted.
(ii) The case $d = 2$ can be interpreted as the complex scalar Gaussian channel in which the real and imaginary parts of the noise are independent and have variance $\sigma^2/2$. In this case, $C(A)$ has units of bits per complex number transmitted.

3. Recall that a discrete memoryless broadcast channel that takes an input $x \in \mathsf{X}$ and generates outputs $y \in \mathsf{Y}$ and $z \in \mathsf{Z}$ is completely determined by the two transition probabilities $w(y|x)$ and $\tilde{w}(z|x)$. The binary symmetric broadcast channel arises when w is a binary symmetric channel (BSC) with crossover probability ε and $\tilde{w}$ is a BSC with crossover probability $\tilde{\varepsilon}$ as in the following diagram:



$w(y|x)$ $\qquad$ $\tilde{w}(z|x)$

Assuming $0 \le \varepsilon < 1/2$ and $\varepsilon \le \tilde{\varepsilon}$, show that this channel is always degraded; i.e., find the value of α so that the cascade



$w(y|x)$ $\qquad$ $q(z|y)$

is such that if $P_{XYZ}(x,y,z) = q(z|y)w(y|x)p(x)$ for any $p(x)$, then $\mathsf{P}(Z=1|X=0) = \mathsf{P}(Z=0|X=1) = \tilde{\varepsilon}$.

4. Consider the following rate-distortion problem. Let $\mathsf{X} = \{0,1\}$ and $\mathsf{Y} = \{0,1,2\}$. Define the distortion function $d(x,y)$ by

$$\begin{aligned} d(0,0) &= d(1,1) = 0 \\ d(0,2) &= d(1,2) = 1 \\ d(0,1) &= d(1,0) = \infty. \end{aligned}$$

In other words, there is no cost for representing $x = 0$ by $y = 0$ or $x = 1$ by $y = 1$. There is cost one for representing either $x = 0$ or $x = 1$ by $y = 2$. There is infinite cost for representing $x = 0$ by $y = 1$ or $x = 1$ by $y = 0$. If the source pmf is $p(0) = p(1) = 1/2$, find

$$R(D) = \inf_{W:\mathsf{E}[d(X,Y)] \le D} I(X \wedge Y),$$

where it is understood that $P_{XY}(x,y) = p(x)W(y|x)$. *Hint:* Start by computing $\mathsf{E}[d(X,Y)]$. Use the convention

$$\lambda \cdot \infty = \begin{cases} 0, & \lambda = 0, \\ \infty, & \lambda > 0. \end{cases}$$