• Exponential density with parameter $\mu$: $f(\gamma) = \mu e^{-\mu \gamma}, \gamma \geq 0$.

• The binary entropy function: $h(\theta) := -[\theta \log \theta + (1 - \theta) \log(1 - \theta)]$. 
1. **Background:** Consider a discrete-time, additive white Gaussian noise channel subject to block fading. This means that when a codeword \((x_1, \ldots, x_n) \in \mathbb{R}^n\) is transmitted, the \(k\)th received channel output symbol is \(Y_k = \sqrt{G}x_k + Z_k\), where the \(Z_k\) are i.i.d. zero-mean Gaussians with common variance \(\sigma^2\). If the receiver knows the value of \(G\), then the codeword constraint
\[
\frac{1}{n} \sum_{k=1}^{n} x_k^2 \leq A \quad \text{is equivalent to} \quad \frac{1}{n} \sum_{k=1}^{n} (\sqrt{G}x_k)^2 \leq GA,
\]
and so the capacity is \(\frac{1}{2} \log(1 + GA/\sigma^2)\). If we put
\[
C(\gamma) := \frac{1}{2} \log(1 + \gamma),
\]
then the capacity is \(C(GA/\sigma^2)\). However, since the fading coefficient \(\sqrt{G}\) is random, the signal-to-noise ratio (SNR) \(\Gamma := GA/\sigma^2\) is also random. If the SNR falls below a threshold \(\gamma\), we say that an **outage** occurs. The parametric curve
\[
\left( P(\Gamma \leq \gamma), C(\gamma) \right), \quad \gamma \geq 0,
\]
expresses the capacity \(c = C(\gamma)\) as a function of the **outage probability** \(p = P(\Gamma \leq \gamma)\).

**Question:** Find a formula to express the capacity \(c\) as a function of the outage probability \(p\) if \(\Gamma\) has an exponential density with parameter \(\mu\) (Rayleigh fading). Your formula should involve only \(p\) and \(\mu\).
2. Consider the $d$-dimensional, discrete-time Gaussian channel defined as follows. The response of the channel to an input symbol $x \in \mathbb{R}^d$ is

$$Y = x + Z,$$

where $Z$ is a $d$-dimensional, zero-mean Gaussian random vector with i.i.d. components such that $E[\|Z\|^2] = \sigma^2$. In other words, $Z$ has covariance matrix $(\sigma^2/d)I$, where $I$ is the $d$-dimensional identity matrix. The channel is subject to the following cost constraint. Every codeword $x = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ must satisfy

$$\frac{1}{n} \sum_{k=1}^n \|x_k\|^2 \leq A.$$

The capacity of this channel is

$$C(A) = \max_{X \in \mathcal{C}} I(X \wedge Y),$$

where it is understood that $Y = X + Z$ with $X$ and $Z$ independent. Find $C(A)$ in terms of $A$, $\sigma^2$, and $d$.

Remarks: (i) When the base of the logarithm is two, the capacity has units of bits per channel use. However, since each channel use transmits $d$ real numbers, $C(A)/d$ is the capacity in bits per real number transmitted.

(ii) The case $d = 2$ can be interpreted as the complex scalar Gaussian channel in which the real and imaginary parts of the noise are independent and have variance $\sigma^2/2$. In this case, $C(A)$ has units of bits per complex number transmitted.
3. Recall that a discrete memoryless broadcast channel that takes an input $x \in X$ and generates outputs $y \in Y$ and $z \in Z$ is completely determined by the two transition probabilities $w(y|x)$ and $\tilde{w}(z|x)$. The binary symmetric broadcast channel arises when $w$ is a binary symmetric channel (BSC) with crossover probability $\epsilon$ and $\tilde{w}$ is a BSC with crossover probability $\tilde{\epsilon}$ as in the following diagram:

\[
\begin{array}{c}
0 \\
1 - \epsilon \\
\epsilon \\
1 \\
\end{array}
\begin{array}{c}
0 \\
1 - \epsilon \\
\epsilon \\
1 \\
\end{array}
\begin{array}{c}
1 - \epsilon \\
\epsilon \\
0 \\
1 \\
\end{array}
\begin{array}{c}
1 - \epsilon \\
\epsilon \\
0 \\
1 \\
\end{array}
\]

$w(y|x)$

$\tilde{w}(z|x)$

Assuming $0 \leq \epsilon < 1/2$ and $\epsilon \leq \tilde{\epsilon}$, show that this channel is always degraded; i.e., find the value of $\alpha$ so that the cascade

\[
\begin{array}{c}
0 \\
1 - \alpha \\
\alpha \\
1 \\
\end{array}
\begin{array}{c}
0 \\
1 - \alpha \\
\alpha \\
1 \\
\end{array}
\begin{array}{c}
1 - \alpha \\
\alpha \\
0 \\
1 \\
\end{array}
\begin{array}{c}
1 - \alpha \\
\alpha \\
0 \\
1 \\
\end{array}
\]

$w(y|x)$

$q(z|y)$

is such that if $P_{XYZ}(x,y,z) = q(z|y)w(y|x)p(x)$ for any $p(x)$, then $P(Z = 1|X = 0) = P(Z = 0|X = 1) = \tilde{\epsilon}$. 
4. Consider the following rate-distortion problem. Let $X = \{0, 1\}$ and $Y = \{0, 1, 2\}$. Define the distortion function $d(x, y)$ by
\[
\begin{align*}
d(0, 0) &= d(1, 1) = 0 \\
d(0, 2) &= d(1, 2) = 1 \\
d(0, 1) &= d(1, 0) = \infty.
\end{align*}
\]
In other words, there is no cost for representing $x = 0$ by $y = 0$ or $x = 1$ by $y = 1$. There is cost one for representing either $x = 0$ or $x = 1$ by $y = 2$. There is infinite cost for representing $x = 0$ by $y = 1$ or $x = 1$ by $y = 0$. If the source pmf is $p(0) = p(1) = 1/2$, find
\[
R(D) = \inf_{W: \mathbb{E}[d(X, Y)] \leq D} I(X \land Y),
\]
where it is understood that $P_{XY}(x, y) = p(x)W(y|x)$. Hint: Start by computing $\mathbb{E}[d(X, Y)]$. Use the convention
\[
\lambda \cdot \infty = \begin{cases} 0, & \lambda = 0, \\ \infty, & \lambda > 0. \end{cases}
\]