ECE 729, Lec. 1
Exam 1
28 February 1995

100 Points

Justify your answers! **Be precise!**

Closed book. Closed Notes. No Calculators.

1. [20 pts.] Let $\mathsf{X} = \{1,2,3,4\}$, and suppose $\mathsf{P}(X=1) = 1/3, \mathsf{P}(X=2) = 1/3, \mathsf{P}(X=3) = 1/4$, and $\mathsf{P}(X=4) = 1/12$. Construct two different Huffman codes, and for each one, compute its expected length.

2. [10 pts.] Let X, Y, and Z be discrete random variables. Show that $H(Z|X) \geq H(Z|X,Y)$.

3. [10 pts.] Let p, q, and r be pmfs on a finite set X. Show that

$$\sum_{x \in \mathsf{X}} p(x) \log \frac{q(x)}{r(x)} \leq \sum_{x \in \mathsf{X}} p(x) \log \frac{p(x)}{r(x)}.$$

4. [20 pts.] Let X be a discrete random variable, and let $U = f(X)$, where f is a deterministic function. Show that $H(U) \leq H(X)$.
   Hint: You don't need to use the inequality $\log t \leq (\log e)(t-1)$.

5. Let $\mathsf{X} = \mathsf{Y} = \{1,2,3\}$, and let

$$\begin{aligned} W(\cdot|1) &= (\tfrac{2}{3}, \tfrac{1}{3}, 0), \\ W(\cdot|2) &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}), \\ W(\cdot|3) &= (0, \tfrac{1}{3}, \tfrac{2}{3}). \end{aligned}$$

The goal of this problem is to find the capacity of the DMC with transition probability W.

   (a) [25 pts.] Let $P_{X,Y}(x,y) := p(x)W(y|x)$. If $p(2)$ is fixed, show that $H(Y)$ is maximized if $p(1) = p(3)$, and this maximum value does not depend on $p(2)$.

   (b) [15 pts.] Find $\sup_p I(p \times W)$. What is the maximizing distribution p?