ECE 729, Lec. 1
Final Exam
Due Wednesday, 12:05 pm, 14 May 1997

100 Points

Justify your answers! **Be precise!**

Open Class Notes.
Do NOT use any books.
Do NOT discuss the exam with anyone.

1. Let $X = [X_1, \ldots, X_n]'$ be a zero-mean, Gaussian random vector with covariance matrix $R = \mathsf{E}[XX']$. In other words,
$$R_{ij} \;=\; \mathsf{E}[X_i X_j].$$
The density of X is
$$f(x) \;=\; \frac{\exp(-\frac{1}{2}x'R^{-1}x)}{(2\pi)^{n/2}\sqrt{\det R}},$$
where $\det R$ is the determinant of R.
   (a) [15 pts.] Calculate the differential entropy of X and show that it is given by
$$h(X) \;=\; \tfrac{1}{2}\log[(2\pi e)^n \det R].$$
   (b) [10 pts.] For the $n = 2$ case, write $X = [Y, Z]'$ and
$$R \;=\; \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$$
   Calculate $I(Y \wedge Z)$ in terms of α, β, and γ. (*Remark.* It is interesting to evaluate $I(Y \wedge Z)$ in the special case $\alpha = \beta = r > 0$ and $\gamma = r\rho$ for some $-1 < \rho < 1$.)
2. [10 pts.] Let X be a real-valued, zero-mean, Gaussian random variable with variance σ^2. Consider the 1-bit quantizer,
$$\hat{X} \;:=\; aI_{(-\infty,0]}(X) + bI_{(0,\infty)}(X).$$
Find a and b (in terms of σ) to minimize $\mathsf{E}[|X - \hat{X}|^2]$. **Evaluate all integrals.**
3. [15 pts.] Let (R_k, D_k) be a sequence of achievable rate-distortion pairs. Suppose that $(R_k, D_k) \to (R, D)$. Prove that (R, D) is an achievable rate-distortion pair.
4. [15 pts.] Let $\{X_k\}$ be a discrete, stationary source with entropy rate
$$\mathscr{H}(\{X_k\}) \;:=\; \lim_{n\to\infty} \frac{1}{n} H(X_1, \ldots, X_n).$$
Show that for any fixed $m \geq 0$,
$$\lim_{n\to\infty} \frac{1}{n} H(X_n, \ldots, X_1 | X_0, X_{-1}, \ldots, X_{-m}) \;=\; \mathscr{H}(\{X_k\}).$$
5. [15 pts.] Let $\{X_k\}$ be a discrete, stationary source. In addition, assume that the source is Markov, i.e.,
$$\mathsf{P}(X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_1 = x_1) \;=\; \mathsf{P}(X_n = x_n | X_{n-1} = x_{n-1}).$$
By stationarity, $p(y|x) := \mathsf{P}(X_n = y | X_{n-1} = x)$ does not depend on n. Similarly, $q(x) := \mathsf{P}(X_n = x)$ does not depend on n. Express $\mathscr{H}(\{X_k\})$ in terms of q and p.
6. [20 pts.] Determine the capacity region of the DM MAC whose single-letter output is $Z = X \cdot Y$, where X and Y take values in the set $\{-1, +1\}$.