ECE 730, Lec. 1
Exam 2
Wednesday, 11 Dec. 2002
5–7 pm in 159 ME

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring two sheets of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

1. [15 pts.] The Gaussian signal $X \sim N(0, \sigma^2)$ is subjected to independent Rayleigh fading so that the received signal is $Y = ZX$, where $Z \sim \text{Rayleigh}(1)$ and X are independent. Find the density of Y. *Hint:* First find the moment generating function of Y.

2. [10 pts.] Find the stationary distribution of the continuous-time Markov chain with generator matrix
$$G = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -5 & 3 \\ 5 & 4 & -9 \end{bmatrix}.$$

3. [15 pts.] Let $X_n \sim \text{Cauchy}(1/n)$. Determine whether or not X_n converges in probability to zero. **Justify your answer.**

4. Let X_n converge almost surely to X.

   (a) [10 pts.] Show that
   $$\frac{1}{1+X_n^2} \quad \text{converges almost surely to} \quad \frac{1}{1+X^2}.$$

   (b) [10 pts.] Determine whether or not
   $$\lim_{n\to\infty} \mathsf{E}\left[\frac{1}{1+X_n^2}\right] = \mathsf{E}\left[\frac{1}{1+X^2}\right].$$

   Justify your answer.

5. [10 pts.] Let $X_n \sim \text{Laplace}(n)$. Show that X_n converges almost surely to zero.

6. [15 pts.] Let f be a given function that satisfies $\int_0^\infty f(t)^2\,dt < \infty$, and let W_t be a Wiener process with $\mathsf{E}[W_t^2] = t$. Find the projection of
$$\int_0^\infty f(t)\,dW_t$$
onto the subspace
$$M := \left\{ \int_0^1 g(t)\,dW_t : \int_0^1 g(t)^2\,dt < \infty \right\}.$$

7. [15 pts.] Suppose state j of a discrete-time Markov chain is “transient” in the sense that
$$\sum_{n=1}^\infty p_{ij}^{(n)} < \infty,$$
where $p_{ij}^{(n)} := \mathsf{P}(X_n = j | X_0 = i)$. Show that
$$\mathsf{P}\left(\bigcap_{N=1}^\infty \bigcup_{n=N}^\infty \{X_n = j\} \middle| X_0 = i \right) = 0.$$