ECE 730, Lec. 1
Final Exam
Tuesday, 16 Dec. 2003
10:05 am – 12:05 pm in 2535 EH

100 Points

Justify your answers! **Be precise!**

Open Book **Open Notes** **Closed Neighbor**

No Calculators

1. [15 pts.] Let W_t be a standard Wiener process, and let the random variable $T \sim \exp(\lambda)$ be independent of the Wiener process. Define the random variable
$$Y := \int_0^T \tau^n \, dW_\tau.$$
Evaluate $\mathsf{E}[Y^2]$. Your final answer should not involve any integrals.

2. [10 pts.] Let X_n have the density
$$f_n(x) = \frac{(p-1)}{n^{p-1}} x^{-p}, \quad x \geq 1/n,$$
for some fixed $p > 2$. Show that X_n converges almost surely to zero.

3. [15 pts.] Let $Y_0, Y_1, \ldots$ be a sequence of continuous random variables with conditional densities that satisfy the density version of the Markov property,
$$f_{Y_{n+1}|Y_n \cdots Y_0}(y_{n+1}|y_n, \ldots, y_0) = f_{Y_{n+1}|Y_n}(y_{n+1}|y_n).$$
Further assume that the transition density $f_{Y_{n+1}|Y_n}(z|y) = p(z|y)$ does not depend on n. Put $X_0 \equiv 0$, and $n \geq 1$, put
$$X_n := \sum_{k=1}^{n} W_k, \quad \text{where} \quad W_k := Y_k - \int_{-\infty}^{\infty} z p(z|Y_{k-1}) \, dz.$$
Determine whether or not X_n is a martingale with respect to $\{Y_n\}$; i.e., determine whether or not $\mathsf{E}[X_{n+1}|Y_n, \ldots, Y_0] = X_n$.

4. [20 pts.] Let $f(x)$ be a probability density function. Let X_n have density $f_n(x) = n f(nx)$. Determine whether or not X_n converges in probability to zero. **Justify your answer.**

5. [20 pts.] Let X_n converge in mean of order 2 to X. Determine whether or not
$$\lim_{n\to\infty} \mathsf{E}\left[X_n^2 e^{-X_n^2}\right] = \mathsf{E}\left[X^2 e^{-X^2}\right].$$
Justify your answer.

6. [20 pts.] Find two different stationary distributions
$$\pi = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \quad \text{and} \quad \tilde{\pi} = \begin{bmatrix} \tilde{\pi}_0 & \tilde{\pi}_1 & \tilde{\pi}_2 & \tilde{\pi}_3 \end{bmatrix}$$
for the 4-state Markov chain with the following state transition diagram.