ECE 730, Lec. 1
Final Exam
Saturday, 18 Dec. 2004
12:25 pm – 2:25 pm in 3534 EH

100 Points

Justify your answers! **Be precise!**

You may bring two sheets of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

1. [20 pts.] Let X_t be a zero-mean process with continuous correlation function $R(t,s)$ satisfying $\int_0^T R(t,s)\varphi_k(s) = \lambda_k \varphi_k(t)$. The Karhunen–Loève expansion of X_t is

$$X_t = \sum_{k=1}^{\infty} A_k \varphi_k(t),$$

where $A_k = \int_0^T X_s \varphi_k(s)\,ds$ and the A_k are uncorrelated with $\mathsf{E}[A_k^2] = \lambda_k$. Let M_L denote the subspace of random variables spanned by $A_1, \ldots, A_L$; i.e.,

$$M_L = \left\{ \sum_{i=1}^{L} c_i A_i : \text{the } c_i \text{ are nonrandom} \right\}.$$

For fixed t, find the projection of X_t onto M_L.

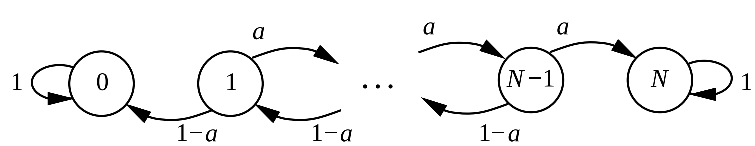
2. [15 pts.] Let $Y_n \sim \text{Bernoulli}(p_n)$, and put $X_n := X + n^2(-1)^n Y_n$, where $X \sim N(0,1)$. Determine whether or not there is a sequence p_n such that X_n converges almost surely to X but not in mean. **Justify your answer.**

3. [15 pts.] Let U be a uniform$[0,1]$ random variable that is independent of a Poisson process N_t with rate $\lambda = 1$. Put

$$Y_t := N_{\ln(1+tU)}.$$

Find the probability generating function of Y_t, $G(z) := \mathsf{E}[z^{Y_t}]$ for real z.

4. [15 pts.] Let X_n be a Markov chain with the following state transition diagram where $0 < a < 1$.



Assume that $\mathsf{P}(X_0 = i_0) = 1$ for some $0 < i_0 < N$. Put $\rho := (1-a)/a$, and define $Y_n := \rho^{X_n}$. Determine whether or not Y_n is a martingale with respect to X_n. In other words, determine whether or not

$$\mathsf{E}[Y_{n+1} | X_n, \ldots, X_0] = Y_n, \quad n = 1, 2 \ldots.$$

Justify your answer.

5. [15 pts.] Let W_t be a standard Wiener process, and let $\int_0^\infty g(\tau)^2\,d\tau < \infty$. Put

$$X_t := \int_0^t g(\tau)\,dW_\tau.$$

We know that X_t is a Gaussian process. Does it have independent increments? **Justify your answer.**

6. [20 pts.] Let $X_n \sim N(0, 1/n^2)$ and $Y_n \sim \exp(n)$. Determine whether or not $X_n - Y_n$ converges in distribution to zero. **Justify your answer.**