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ECE 730 Exam 1 Solutions
Spring 2011

1) $$M_X(s) := E[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} e^{x-e^x}\, dx$$

$$= \int_{-\infty}^{\infty} (e^x)^s e^{-e^x} \cdot e^x\, dx \qquad t = e^x,\quad dt = e^x dx$$

$$= \int_0^{\infty} t^s e^{-t}\, dt$$

$$= \int_0^{\infty} t^{(s+1)-1} e^{-t}\, dt$$

$$= \Gamma(s+1), \quad re(s+1) > 0 \text{ or } \underline{\underline{re(s) > -1}}.$$

<u>Remark.</u> $\exp[-x - e^{-x}]$ is a special case of the Gumbel density, which arises in extreme-value theory.

2) $$X(\omega) = I_{\{1,2\}}(\omega) - \omega I_{\{3,4\}}(\omega)$$

$$= \begin{cases} 1, & \omega \in \{1,2\}, \\ -3, & \omega \in \{3\}, \\ -4, & \omega \in \{4\}, \end{cases}$$

and so

$$E[X] = 1 \cdot P(\{1,2\}) - 3P(\{3\}) - 4P(\{4\}).$$

But, since $\{3\}, \{4\} \notin \mathcal{A}$, these last two probabilities are not defined! $\therefore$ $E[X]$ is not defined.

Alternatively, you can show directly that X is not a RV since

$$\{\omega \in \Omega : X(\omega) < -3\} = \{4\} \notin \mathcal{A}.$$

3) Observe that for fixed $1 \le y \le 2$, $f_{XY}(x,y) = y^2 e^{-y^2 x}$ is an $\exp(\lambda = y^2)$ density in $x \ge 0$. $\therefore$ $f_{X|Y}(\cdot|y) \sim \exp(\lambda = y^2)$. So,

$$E[X^3 Y^2 | Y=y] = E[X^3 y^2 | Y=y] = y^2 E[X^3 | Y=y]$$

$$= y^2 \cdot \frac{3!}{(y^2)^3}$$ from the table attached to the exam.

Next, using the law of total prob.,

$$E[X^3 Y^2] = E[\,E[X^3 Y^2 | Y]\,]$$

$$= E\left[\frac{3!}{Y^4}\right] = 3! \int_1^2 y^{-4}\,dy = \frac{3!}{-3} y^{-3}\Big|_1^2 = 2\left(1 - \frac{1}{8}\right)$$

$$= 2 \cdot 7/8 = 7/4.$$

4) Let $Y := P'X$ be the decorrelating transform for X; i.e., $P'CP = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. So Y_i are iid $N(0, \lambda_i)$. Put $I := \{i : \lambda_i > 0\}$. Then $Y_i \equiv 0$ for $i \notin I$, and so

$$\|X\|^2 = X'X = (PY)'(PY) = Y'P'PY = \|Y\|^2 = \sum_{i \in I} Y_i^2.$$

Next,

$$M(s) = E[e^{s\|X\|^2}] = E[e^{s\|Y\|^2}] = E[e^{s\sum_{i\in I} Y_i^2}] = \prod_{i \in I} E[e^{s\lambda_i (Y_i/\sqrt{\lambda_i})^2}]$$

Since $Y_i/\sqrt{\lambda_i} \sim N(0,1)$, $(Y_i/\sqrt{\lambda_i})^2 \sim$ chi-squared with one degree of freedom. So,

$$M(s) = \prod_{i \in I} M_{(Y_i/\sqrt{\lambda_i})^2}(s\lambda_i) = \prod_{i \in I} \left(\frac{1/2}{1/2 - s\lambda_i}\right)^{1/2}$$ from table attached to the exam

$$= \prod_{i \in I} \left(\frac{1}{1 - 2s\lambda_i}\right)^{1/2} = \prod_{i=1}^{n} \left(\frac{1}{1 - 2s\lambda_i}\right)^{1/2}.$$

If C is invertible, which you were not to assume, then here is another solution:

$$M(s) = E[e^{s\|X\|^2}] = \int_{\mathbb{R}^n} e^{sx'x} \cdot \frac{e^{-x'C^{-1}x/2}}{(2\pi)^{n/2}\sqrt{\det C}}\, dx$$

$$= \frac{1}{\sqrt{\det C}} \underbrace{\int \frac{e^{-x'[-2sI + C^{-1}]x/2}}{(2\pi)^{n/2}\sqrt{\det R}}\, dx}_{=1} \sqrt{\det R},$$

where $R := [-2sI + C^{-1}]^{-1}$.

So

$$M(s) = \sqrt{\frac{\det R}{\det C}} = \sqrt{\frac{1}{\det R^{-1}}\,\frac{1}{\det C}} = \frac{1}{\sqrt{\det R^{-1}C}}.$$

Now

$$R^{-1}C = [-2sI + C^{-1}]C = -2sC + I = I - 2sC = I - 2s P\Lambda P' = P(I - 2s\Lambda)P',$$

and so $\det R^{-1}C = \det P \det P' \det(I - 2s\Lambda) = \det(PP')\det(I - 2s\Lambda)$
$= 1 \cdot (1-2s\lambda_1)\cdots(1-2s\lambda_n)$. Thus,

$$M(s) = \frac{1}{\sqrt{(1-2s\lambda_1)\cdots(1-2s\lambda_n)}}.$$

5) Let X_i = amount of snow of ith storm.

let $Y_i := I_{(t,\infty)}(X_i) = \begin{cases} 1 & \text{if } i\text{th storm drops} > t \text{ inches.} \\ 0 & \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \leq t \quad \text{"} \end{cases}$

Put $S_n := Y_1 + \cdots + Y_n$. Since the X_i are iid., so are the Y_i.
& $P(Y_i = 1) = P(X_i > t) = e^{-\lambda t}$ from the table. Since S_n is the sum of iid Bernoulli$(e^{-\lambda t})$, $S_n \sim$ binomial$(n, e^{-\lambda t})$. $\therefore$

$$P(S_n = k) = \binom{n}{k}(e^{-\lambda t})^k(1 - e^{-\lambda t})^{n-k}.$$