ECE 730
Final Exam
13 May 2011
5:05 pm – 7:05 pm in 2534 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring two sheets of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

1. Let Y_t be the response of an LTI system to a zero-mean wide-sense stationary (WSS) process X_t. If the input X_t has correlation function

$$R_X(\tau) = 2\left[\frac{\sin(2\pi\tau)}{2\pi\tau}\right]^2,$$

and the LTI system has transfer function

$$H(f) := \begin{cases} \dfrac{1}{\sqrt{1-|f|/2}}, & |f| \le 1, \\ 0, & |f| > 1, \end{cases}$$

find the correlation function of Y_t.

2. Suppose X_n and X are random variables with zero means, unit variances, and correlation $\mathsf{E}[X_n X] = 1 - 1/(2n^2)$.

   (a) Show that X_n converges in mean of order 2 to X.
   (b) Show that X_n converges almost surely to X.

3. Let X_t and Y_t be zero-mean, jointly wide-sense stationary (J-WSS) random processes with given cross-correlation function $R_{XY}(\tau)$ and given autocorrelation functions $R_X(\tau)$ and $R_Y(\tau)$. Let $Z_t := \operatorname{sgn}(Y_t)$, where sgn denotes the sign function defined by

$$\operatorname{sgn}(y) := \begin{cases} 1, & y > 0, \\ 0, & y = 0, \\ -1, & y < 0. \end{cases}$$

Assuming X_t and Y_t are jointly Gaussian processes, find the cross-correlation $\mathsf{E}[X_{t_1} Z_{t_2}]$. **Evaluate all integrals.**

4. Let $X_n \sim \text{geometric}_0(p_n)$, $0 \le p_n < 1$, and suppose that X_n converges in mean of order 2 to some $X \in L^2$. Find the distribution of X assuming $\mathsf{var}(X) > 0$. **Justify your answer.**

5. Let $\{N_t, t \ge 0\}$ be a Poisson process with intensity $\lambda = 1$. Define a new process $M_t := N_{v(t)}$, where $v\colon [0,\infty) \to [0,\infty)$ is an invertible function that is strictly increasing, differentiable, and satisfies $v(0) = 0$. The occurrence times of M_t are

$$S_n := \min\{t : M_t \ge n\}.$$

Find the density of S_n.