ECE 730
Final Exam Sol.

Sp, 2011

① $S_X(f) = (1 - |f|/2)\, I_{[-2,2]}(f)$

$S_Y(f) = |H(f)|^2 S_X(f)$

$= \dfrac{1}{1 - |f|/2}(1 - |f|/2)\, I_{[-1,1]}(f)$

$= I_{[-1,1]}(f)$

$\therefore R_Y(\tau) = 2\dfrac{\sin(2\pi\tau)}{2\pi\tau} = \sqrt{2}\sqrt{R_X(f)}$.

② (a) $E[|X_n - X|^2] = E[X_n^2] - 2E[X_n X] + E[X^2]$

$= 1 \quad -2(1 - 1/(2n^2)) + 1$

$= \frac{1}{n^2} \to 0.$

$\therefore X_n \xrightarrow{L^2} X.$

(b) $\displaystyle\sum_{n=1}^{\infty} P(|X_n - X| \geq \varepsilon) \leq \sum_{n=1}^{\infty} \frac{E[|X_n - X|^2]}{\varepsilon^2}$, by Markov's ineq.

$\displaystyle = \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$

$\therefore X_n \to X$ a.s.

③ $E[X_{t_1} Z_{t_2}] = E[X_{t_1}\,\mathrm{sgn}(Y_{t_2})]$

$= E[\,E[X_{t_1}\,\mathrm{sgn}(Y_{t_2}) \mid Y_{t_2}]\,]$

$= E[\,E[X_{t_1} \mid Y_{t_2}]\,\mathrm{sgn}(Y_{t_2})\,]$

$= E[\,A Y_{t_2}\,\mathrm{sgn}(Y_{t_2})\,]$, $\quad A\, C_{Y_{t_2}} = C_{X_{t_1} Y_{t_2}}$

$A = \dfrac{R_{XY}(t_1 - t_2)}{R_Y(0)}$

Next,

$E[Y_{t_2}\,\mathrm{sgn}(Y_{t_2})] = \int y\,\mathrm{sgn}(y)\, f_Y(y)\, dy$

$= \int |y|\, f(y)\, dy$

$$= 2\int_0^\infty y\,\frac{e^{-(y/\sigma)^2/2}}{\sqrt{2\pi}\,\sigma}\,dy \qquad \sigma^2 = R_Y(0)$$

$$= \frac{2\sigma}{\sqrt{2\pi}}\cdot\left(-e^{-(y/\sigma)^2/2}\right)\Big|_0^\infty$$

$$= \frac{2\sigma}{\sqrt{2\pi}} = \frac{2\,R_Y(0)^{1/2}}{\sqrt{2\pi}}.$$

So,

$$R_{XZ}(t_1,t_2) = \sqrt{\frac{2}{\pi R_Y(0)}}\,R_{XY}(t_1-t_2).$$

(4) As we know, $X_n \xrightarrow{L^2} X \Rightarrow E[X_n^2] \to E[X^2]$ and $E[X_n] \to E[X]$.

$$\therefore \quad \frac{p_n}{1-p_n} = E[X_n] \to E[X] \qquad \boxed{1}$$

and

$$\frac{p_n}{(1-p_n)^2} + \left[\frac{p_n}{1-p_n}\right]^2 = E[X_n^2] \to E[X^2].$$

By $\boxed{1}$, $\left(\frac{p_n}{1-p_n}\right)^2 \to E[X]^2 \qquad \boxed{2}$

& So $\frac{p_n}{(1-p_n)^2} \to \mathrm{var}(X). \qquad \boxed{3}$

Then $\boxed{2}$ & $\boxed{3}$ $\Rightarrow$ $\boxed{p_n = \frac{\left(\frac{p_n}{1-p_n}\right)^2}{\frac{p_n}{(1-p_n)^2}} \to \frac{E[X]^2}{\mathrm{var}(X)}.}$

We now have

$$G_{X_n}(z) = \frac{1-p_n}{1-zp_n} \to \frac{1-p}{1-zp}, \quad p := \frac{E[X]^2}{\mathrm{var}(X)}.$$

$$\therefore X \sim \mathrm{geo}_0(p).$$

⑤ Let's identify $\{t : M_t \geq n\} = \{t : N_{v(t)} \geq n\} = \{t : T_n \leq v(t)\}$, where T_n is the nth occurrence time of $\{N_t, t \geq 0\}$. So,

$$\{t : M_t \geq n\} = \{t : v^{-1}(T_n) \leq t\}$$
$$= [v^{-1}(T_n), \infty).$$

$\therefore$ $S_n = \min[v^{-1}(T_n), \infty) = v^{-1}(T_n)$.

Next,

$$F_{S_n}(s) = P(S_n \leq s) = P(v^{-1}(T_n) \leq s) = P(T_n \leq v(s))$$
$$= F_{T_n}(v(s))$$

$\therefore$

$$f_{S_n}(s) = f_{T_n}(v(s))\, v'(s).$$

Since $T_n \sim \text{Erlang}(n, 1)$,

$$f_{S_n}(s) = \frac{v(s)^{n-1} e^{-v(s)}}{(n-1)!} v'(s)$$