ECE 730 Exam 1 Solutions

Fall 2012

1) The answer is "yes." To explain why, we argue as follows. In order that $\mathcal{P}(\{1,4\})$, $\mathcal{P}(\{2\})$, and $\mathcal{P}(\{3\})$ be defined, it is necessary than any proposed σ-algebra contain $\{1,4\}$, $\{2\}$, and $\{3\}$. Since a σ-algebra is closed under complements, the σ-algebra must also contain $\{1,4\}^c=\{2,3\}$, $\{2\}^c=\{1,3,4\}$, and $\{3\}^c=\{1,2,4\}$. The σ-algebra must also contain all unions and intersections of the preceding 6 sets, as well as $\emptyset$ and $\Omega=\{1,2,3,4\}$. The collection

$$\mathcal{A}:=\{\emptyset,\{2\},\{3\},\{1,4\},\{2,3\},\{1,3,4\},\{1,2,4\},\Omega\}$$

contains the required sets, and since $\mathcal{A}$ contains $\emptyset$, and is closed under complements and unions, $\mathcal{A}$ is therefore a σ-algebra. Note that $\mathcal{A}\neq\{\text{all subsets of }\Omega\}$ since $\{4\}\notin\mathcal{A}$.

2) To begin, use the Law of Total Probability to write

$$E[Y^n\cos(X)]=\int_0^\infty E[Y^n\cos(X)|Y=y]f_Y(y)\,dy$$
$$=\int_0^\infty y^n E[\cos(X)|Y=y]f_Y(y)\,dy.$$

Next,

$$E[\cos(X)|Y=y]=\operatorname{Re}E[e^{jX}|Y=y]=\operatorname{Re}E[e^{j\nu X}|Y=y]\Big|_{\nu=1}$$
$$=\operatorname{Re}e^{-y|\nu|}\Big|_{\nu=1}=e^{-y}.$$

Alternatively, $E[\cos(X)|Y=y]=E\left[\frac{e^{jX}+e^{-jX}}{2}\Big|Y=y\right]$

$$=\tfrac{1}{2}E[e^{j\nu X}|Y=y]\Big|_{\nu=1}+\tfrac{1}{2}E[e^{-j\nu X}|Y=y]\Big|_{\nu=1}$$
$$=\tfrac{1}{2}e^{-y|\nu|}\Big|_{\nu=1}+\tfrac{1}{2}e^{-y|-\nu|}\Big|_{\nu=1}=e^{-y}.$$

Either way, we can now write

$t = 2y \quad dt = 2dy$

$$E[Y^n \cos(X)] = \int_0^\infty y^n e^{-y} e^{-y} dy = \int_0^\infty y^n e^{-2y} dy = \int_0^\infty (t/2)^n e^{-t} \frac{dt}{2}$$

$$= \frac{1}{2^{n+1}} \int_0^\infty t^{(n+1)-1} e^{-t} dt = \frac{\Gamma(n+1)}{2^{n+1}} = \frac{n!}{2^{n+1}}.$$

3) If X_1 and Y are jointly Gaussian, we can find $E[X_1|Y=y]$ using linear MMSE, i.e., $E[X_1|Y=y] = A(y - m_Y) + m_X$, where $AC_Y = C_{X,Y}$. To show that X_1 and Y are jointly Gaussian, write

$$\begin{bmatrix} X_1 \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & b' & & \end{bmatrix} X.$$

Showing only that Y is Gaussian is not enough!

Since X_1 and $Y = b'X$ have zero mean, $E[X_1|Y=y] = Ay$. We can further write

$$C_Y = E[(b'X)(b'X)'] = b'Cb,$$

and

$$C_{X_1,Y} = E[X_1(b'X)] = E[X_1 \sum_{j=1}^n b_j X_j] = \sum_{j=1}^n C_{1j} b_j.$$

So,

$$A = \sum_{j=1}^n C_{1j} b_j \Big/ b'Cb,$$

and

$$E[X_1|Y=y] = \frac{\sum_{j=1}^n C_{1j} b_j}{b'Cb} y.$$

4) From $AC_Y = C_{XY}$ and $BC_Y = C_{XY}$, $(A-B)C_Y = 0$. Next

$$E[\|AY - BY\|^2] = E[\|(A-B)Y\|^2] = E[tr((A-B)YY'(A-B)')] = tr[\underbrace{(A-B)C_Y}_{=0}(A-B)']$$
$$= 0.$$

5) Put $Z := X^2 + Y^2$. Then X^2 and Y^2 are χ_1^2 & so Z is $\chi_2^2 = \exp(1/2)$.

So,

$$P(\text{alarm triggered}) = P(Z > t) = e^{-t/2}.$$

Alternatively, we can find the density of $Z := X^2 + Y^2$ as follows.

$$\mathbb{P}(Z \le z) = \int_{-\infty}^{\infty} \mathbb{P}(X^2 + Y^2 \le z \mid Y = y) f_Y(y)\, dy$$
$$= \int_{-\infty}^{\infty} \mathbb{P}(X^2 \le z - y^2)\, \varphi(y)\, dy,$$

where we have used substitution and independence, and where φ denotes the standard normal density. Since $\mathbb{P}(X^2 \le z - y^2) = 0$ for $z - y^2 < 0$, i.e., for $|y| > \sqrt{z}$, we can write

$$\mathbb{P}(Z \le z) = \int_{-\sqrt{z}}^{\sqrt{z}} \mathbb{P}(|X| \le \sqrt{z - y^2})\, \varphi(y)\, dy$$
$$= \int_{-\sqrt{z}}^{\sqrt{z}} 2\left[\Phi(\sqrt{z - y^2}) - \Phi(0)\right] \varphi(y)\, dy$$
$$= 4 \int_0^{\sqrt{z}} \left[\Phi(\sqrt{z - y^2}) - \Phi(0)\right] \varphi(y)\, dy,$$

where Φ denotes the standard normal cdf. Next,

$$f_Z(z) = 4\left[\Phi(\sqrt{z - y^2}) - \Phi(0)\right] \varphi(y)\Big|_{y=\sqrt{z}} \cdot \frac{1}{2\sqrt{z}}$$
$$+ 4\int_0^{\sqrt{z}} \varphi(\sqrt{z - y^2}) \cdot \frac{1}{2\sqrt{z - y^2}}\, \varphi(y)\, dy$$
$$= 2\int_0^{\sqrt{z}} \frac{e^{-(z - y^2)/2}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{z - y^2}} \cdot \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$
$$= \frac{e^{-z/2}}{\pi} \int_0^{\sqrt{z}} \frac{1}{\sqrt{z - y^2}}\, dy = \frac{e^{-z/2}}{\pi} \sin^{-1}\left(\frac{y}{\sqrt{z}}\right)\Big|_{y=0}^{y=\sqrt{z}}$$
$$= \frac{e^{-z/2}}{\pi} \sin^{-1}(1) = \frac{1}{2} e^{-z/2}$$
$$\therefore\ Z \sim \exp(1/2)$$

$$\therefore\ \mathbb{P}(Z > t) = e^{-t/2}.$$