ECE 730
Final Exam
21 December 2012
5:05–7:05 pm in 2317 EH

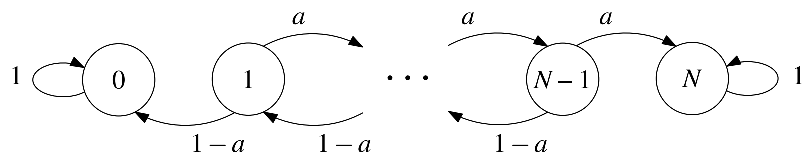
100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring two sheets of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

1. Consider the discrete-time Markov chain with state transition diagram below:



where $0 \le a \le 1$. Are there any values of a for which X_n is a martingale with respect to itself? **Justify your answer.**

Solution. We must determine whether or not $\mathsf{E}[X_{n+1}|X_n,\ldots,X_0] = X_n$. By the Markov property, $\mathsf{E}[X_{n+1}|X_n,\ldots,X_0] = \mathsf{E}[X_{n+1}|X_n]$. To be a martingale, we must show that $\mathsf{E}[X_{n+1}|X_n = i] = i$ for $i = 0,\ldots,N$. First note that

$$\mathsf{E}[X_{n+1}|X_n = 0] = 0 \quad \text{and} \quad \mathsf{E}[X_{n+1}|X_n = N] = N.$$

For $0 < i < N$,

$$\mathsf{E}[X_{n+1}|X_n = i] = \sum_j j p_{ij} = (i+1)a + (i-1)(1-a) = ai + a + i - 1 - ai + a = i + 2a - 1.$$

Putting these two formulas together yields

$$\mathsf{E}[X_{n+1}|X_n = i] = i + (2a-1)I_{\{1,\ldots,N-1\}}(i),$$

which implies

$$\mathsf{E}[X_{n+1}|X_n] = X_n + (2a-1)I_{\{1,\ldots,N-1\}}(X_n).$$

So the only possible value of a is $a = 1/2$.

2. Let X and Y be random variables with $X \in L^1$. Let q be an invertible function, and put $Z := q(Y)$. Put $\widehat{g}(y) := \mathsf{E}[X|Y = y]$. Determine whether or not $\mathsf{E}[X|Z] = \widehat{g}(q^{-1}(Z))$. **Justify your answer.**

Solution. The most direct solution is obtained by writing

$$\begin{aligned}
\mathsf{E}[X|Z] &= \mathsf{E}[X|q(Y)], && \text{since } Z := q(Y),\\
&= \mathsf{E}\big[\mathsf{E}[X|Y]\big|q(Y)\big], && \text{by the smoothing property (13.28)},\\
&= \mathsf{E}[\widehat{g}(Y)|Z], && \text{since } \widehat{g}(Y) = \mathsf{E}[X|Y] \text{ and } Z := q(Y),\\
&= \mathsf{E}[\widehat{g}(q^{-1}(Z))|Z], && \text{since } Y = q^{-1}(Z),\\
&= \widehat{g}(q^{-1}(Z)), && \text{by Problem 13.56.}
\end{aligned}$$

Alternative Solution. The characterizing equation for $\mathsf{E}[X|Z]$ is

$$\mathsf{E}[Xh(Z)] = \mathsf{E}\big[\mathsf{E}[X|Z]h(Z)\big], \quad \text{for every bounded function } h.$$

By uniqueness of conditional expectation, it suffices to show that for all bounded functions $h(z)$,

$$\mathsf{E}[Xh(Z)] = \mathsf{E}[\widehat{g}(q^{-1}(Z))h(Z)].$$

Write

$$\begin{aligned}
\mathsf{E}[Xh(Z)] &= \mathsf{E}[Xh(q(Y))], && \text{since } Z = q(Y),\\
&= \mathsf{E}\big[\mathsf{E}[X|Y]h(q(Y))\big], && \text{by the characterizing equation for } \mathsf{E}[X|Y],\\
&= \mathsf{E}\big[\widehat{g}(Y)h(q(Y))\big], && \text{by definition of } \widehat{g},\\
&= \mathsf{E}\big[\widehat{g}(q^{-1}(Z))h(Z)\big], && \text{since } Y = q^{-1}(Z).
\end{aligned}$$

3. Let X be a zero-mean Gaussian random vector with invertible covariance matrix C. For $t > 0$, put

$$B_t := \{x : x'C^{-1}x > t\}.$$

If X has dimension $2n$, find a simple formula (no integrals) for $\mathsf{P}(X \in B_t)$.

Solution. First observe that $\mathsf{P}(X \in B_t) = \mathsf{P}(\|C^{-1/2}X\|^2 > t)$. Then put $Y := C^{-1/2}X$ so that Y is Gaussian with zero mean and covariance $C^{-1/2}CC^{-1/2} = I$. Thus, $\|Y\|^2$ is chi-squared with $2n$ degrees of freedom; equivalently $\|Y\|^2$ is Erlang$(n, 1/2)$. Hence,

$$\mathsf{P}(\|Y\|^2 > t) = \sum_{k=0}^{n-1} \frac{(t/2)^k}{k!} e^{-t/2}.$$

Alternative Solution. Let $X = PY$ be the Karhunen–Loève expansion of X, where $P'CP = \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_{2n})$. Then

$$\mathsf{P}(X \in B_t) = \mathsf{P}(X'C^{-1}X > t) = \mathsf{P}\big((PY)'C^{-1}(PY) > t\big) = \mathsf{P}(Y'\Lambda^{-1}Y > t) = \mathsf{P}\left(\sum_{k=1}^{2n} \frac{Y_k^2}{\lambda_k} > t\right).$$

Since $Y = P'X$ is Gaussian, so is $\Lambda^{-1/2}Y$. Since $Y_k/\sqrt{\lambda_k} \sim N(0,1)$, this last sum is chi-squared with $2n$ degrees of freedom, and the solution finishes as above.

4. Let N_t be a Poisson process with intensity λ. Put $Y_t := g(N_t)$, where

$$g(x) := \begin{cases} x, & 0 \le x < 1,\\ 1+x, & x \ge 1. \end{cases}$$

For $0 \le s < t < \infty$, compute $\mathsf{E}[g(N_t) - g(N_s)]$.

Solution. The most direct solution is obtained by observing that

$$\mathsf{E}[g(N_t)] = \sum_{n=0}^{\infty} g(n)\mathsf{P}(N_t = n) = \sum_{n=1}^{\infty} (1+n)\mathsf{P}(N_t = n) = [1 - \mathsf{P}(N_t = 0)] + \mathsf{E}[N_t] = 1 - e^{-\lambda t} + \lambda t.$$

Then

$$\mathsf{E}[g(N_t) - g(N_s)] = \mathsf{E}[g(N_t)] - \mathsf{E}[g(N_s)] = e^{-\lambda s} - e^{-\lambda t} + \lambda(t-s).$$

Alternative Solution. To use the law of total probability, we first compute

$$\mathsf{E}[g(N_t) - g(N_s)|N_s = n].$$

Since $s<t$ implies $N_s<N_t$, if $N_s=n\geq 1$, then $N_t\geq 1$ as well. Hence, for $n\geq 1$,

$$\begin{aligned}
\mathsf{E}[g(N_t)-g(N_s)|N_s=n] &= \mathsf{E}[(1+N_t)-(1+N_s)|N_s=n] \\
&= \mathsf{E}[N_t-N_s|N_s=n] \\
&= \mathsf{E}[N_t-N_s|N_s-N_0=n] = \mathsf{E}[N_t-N_s] = \lambda(t-s).
\end{aligned}$$

However, for $n=0$,

$$\begin{aligned}
\mathsf{E}[g(N_t)-g(N_s)|N_s=0] &= \mathsf{E}[g(N_t)|N_s=0] \\
&= \mathsf{E}[g(N_t-N_s)|N_s=0] \\
&= \mathsf{E}[g(N_t-N_s)|N_s-N_0=0] \\
&= \mathsf{E}[g(N_t-N_s)] \\
&= \sum_{k=0}^{\infty} g(k)\mathsf{P}(N_t-N_s=k) \\
&= \sum_{k=1}^{\infty} (1+k)\mathsf{P}(N_t-N_s=k) \\
&= \big\{1-\mathsf{P}(N_t-N_s=0)\big\}+\mathsf{E}[N_t-N_s] \\
&= \big[1-e^{-\lambda(t-s)}\big]+\lambda(t-s).
\end{aligned}$$

We can now write

$$\begin{aligned}
\mathsf{E}[g(N_t)-g(N_s)] &= \sum_{n=0}^{\infty} \mathsf{E}[g(N_t)-g(N_s)|N_s=n]\mathsf{P}(N_s=n) \\
&= \big[1-e^{-\lambda(t-s)}\big]\mathsf{P}(N_s=0)+\lambda(t-s) \\
&= \big[1-e^{-\lambda(t-s)}\big]e^{-\lambda s}+\lambda(t-s) \\
&= e^{-\lambda s}-e^{-\lambda t}+\lambda(t-s).
\end{aligned}$$

5. Let m_n be an arbitrary sequence of real numbers, and let σ_n be an arbitrary sequence of positive numbers. Let X be a Laplace random variable with parameter $\lambda=1$. Define a sequence of random variables $Y_n:=\sigma_n X+m_n$. Assume Y_n converges in mean of order 2 to some random variable Y. Determine whether or not Y is a continuous random variable. **Justify your answer.**

   Solution. It is *not* necessary for Y to be a continuous random variable. To see this, consider the case $m_n=m$ for all n and $\sigma_n=1/n$. Then $Y_n=X/n+m$, and $\mathsf{E}[|Y_n-m|^2]=\mathsf{E}[X^2]/n^2=2/n^2\to 0$. Thus, Y_n converges in mean of order 2 to the constant random variable $Y\equiv m$, which is not a continuous random variable.

   A more complete understanding of what is going on is obtained by using the fact that convergence in mean of order 2 implies convergence in distribution. Hence, $\varphi_{Y_n}(\nu)\to\varphi_Y(\nu)$ for all ν. Since $\varphi_{Y_n}(\nu)=e^{j\nu m_n}/(1+\sigma_n^2\nu^2)$, we would like to say that $\varphi_Y(\nu)=e^{j\nu m}/(1+\sigma^2\nu^2)$. The difficulty is that we first have to show m_n and σ_n both converge to finite limits, which we will call m and σ, respectively. A similar difficulty arises if we observe that

$$F_{Y_n}(y) = \mathsf{P}(Y_n \le y) = \mathsf{P}(\sigma_n X + m_n \le y) = \mathsf{P}\Big(X \le \frac{y - m_n}{\sigma_n}\Big) = F_X\Big(\frac{y-m_n}{\sigma_n}\Big) \qquad (*)$$

and then want to write $F_{Y_n}(y) \to F_X\big((y-m)/\sigma\big)$.

To show that m_n and σ_n converge, we argue as follows. By Example 13.11, $\mathsf{E}[Y_n] \to \mathsf{E}[Y]$, and by Problem 13.22, $\mathsf{E}[Y_n^2] \to \mathsf{E}[Y^2]$. Also, $\mathsf{E}[Y_n] \to \mathsf{E}[Y]$ implies $(\mathsf{E}[Y_n])^2 \to (\mathsf{E}[Y])^2$. Hence,

$$\begin{aligned}\lim_{n\to\infty} \mathsf{E}[Y_n^2] - (\mathsf{E}[Y_n])^2 &= \lim_{n\to\infty} \mathsf{E}[Y_n^2] - \lim_{n\to\infty} (\mathsf{E}[Y_n])^2 \\ &= \mathsf{E}[Y^2] - (\mathsf{E}[Y])^2.\end{aligned}$$

In this particular problem, $\mathsf{E}[Y_n] = m_n$ and $\mathsf{E}[Y_n^2] = 2\sigma_n^2 + m_n^2$. Thus, $m_n \to \mathsf{E}[Y]$, and

$$\mathsf{E}[Y^2] - (\mathsf{E}[Y])^2 = \lim_{n\to\infty} \mathsf{E}[Y_n^2] - (\mathsf{E}[Y_n])^2 = \lim_{n\to\infty} (2\sigma_n^2 + m_n^2) - m_n^2 = \lim_{n\to\infty} 2\sigma_n^2.$$

It follows that $\sigma_n \to \sqrt{\big\{\mathsf{E}[Y^2] - (\mathsf{E}[Y])^2\big\}/2}$. Now that we know these limits exist, we put

$$m := \lim_{n\to\infty} m_n = \mathsf{E}[Y] \quad \text{and} \quad \sigma := \lim_{n\to\infty} \sigma_n = \sqrt{\big\{\mathsf{E}[Y^2] - (\mathsf{E}[Y])^2\big\}/2}.$$

Now, whether we consider $e^{j\nu m}/(1+\sigma^2\nu^2)$ or $F_X\big((y-m)/\sigma\big)$, there are two cases to analyze. First, if $\sigma > 0$, then both formulas tell us that Y has density $f_X\big((x-m)/\sigma\big)/\sigma$, which means that Y is a continuous random variable. On the other hand, if $\sigma = 0$, then $\varphi_Y(\nu) = e^{j\nu m}$ is the characteristic function of the constant random variable $Y \equiv m$. We reach the same conclusion using $(*)$: If $\sigma_n \to 0$, then $(*)$ tells us that $F_{Y_n}(y) \to F_X(\infty) = 1$ if $y > m$ and $F_{Y_n}(y) \to F_X(-\infty) = 0$ if $y < m$, which tells us that $Y_n \equiv m$.