ECE 730
Exam 1
21 October 2013
5:15–6:45 pm in 3345 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring one sheet of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

1. Your friend is learning about probability and is working with the probability space $(\Omega, \mathscr{A}, \mathsf{P})$, where the sample space is $\Omega := \{a,b,c,d\}$, the sigma-algebra is $\mathscr{A} := \big\{\varnothing, \{a,b\}, \{c,d\}, \Omega\big\}$, and the probability measure is given by $\mathsf{P}(\{a,b\}) := 1/3$, and $\mathsf{P}(\{c,d\}) := 2/3$. Determine whether or not it is possible to construct a random variable X on this probability space that satisfies $\mathsf{P}(X=1) = 1/3$, $\mathsf{P}(X=2) = 1/3$, and $\mathsf{P}(X=3) = 1/3$. **Justify your answer.**

   Solution. The answer is "No." To see this, assume otherwise that such a random variable exists. Then $\mathscr{A}$ must contain the three disjoint sets $\{\omega \in \Omega : X(\omega) = 1\}$, $\{\omega \in \Omega : X(\omega) = 2\}$, and $\{\omega \in \Omega : X(\omega) = 3\}$. Since each of these sets must have probability $1/3$, these sets must be nonempty and also not equal to Ω. However, $\mathscr{A}$ contains only two such sets.

2. Let $X \sim N(0,1)$ and $N \sim \text{Poisson}(\lambda)$ be independent random variables. Evaluate

$$\mathsf{E}\left[\int_0^X t^N\,dt\right].$$

   For full credit, simplify your answer as much as possible.

   Solution. We can evaluate this expectation using the law of total probability, either by conditioning on X or conditioning on N. For our first solution, we condition on X. To begin, write

$$\begin{aligned}\mathsf{E}\left[\int_0^X t^N\,dt\right] = \mathsf{E}\left[\frac{X^{N+1}}{N+1}\right] = \int_{-\infty}^{\infty} \mathsf{E}\left[\frac{X^{N+1}}{N+1}\middle| X = x\right] f_X(x)\,dx &= \int_{-\infty}^{\infty} \mathsf{E}\left[\frac{x^{N+1}}{N+1}\middle| X = x\right] f_X(x)\,dx \\ &= \int_{-\infty}^{\infty} \mathsf{E}\left[\frac{x^{N+1}}{N+1}\right] f_X(x)\,dx,\end{aligned}$$

   where we have used substitution and independence. Next,

$$\mathsf{E}\left[\frac{x^{N+1}}{N+1}\right] = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \cdot \underbrace{\mathsf{P}(N=n)}_{\text{Poisson pmf}} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \cdot \frac{\lambda^n e^{-\lambda}}{n!} = \frac{e^{-\lambda}}{\lambda} \sum_{n=0}^{\infty} \frac{(\lambda x)^{n+1}}{(n+1)!} = \frac{e^{-\lambda}}{\lambda}\big[e^{\lambda x} - 1\big].$$

   It now follows that

$$\mathsf{E}\left[\int_0^X t^N\,dt\right] = \int_{-\infty}^{\infty} \frac{e^{-\lambda}}{\lambda}\big[e^{\lambda x} - 1\big] f_X(x)\,dx = \frac{e^{-\lambda}}{\lambda}\big\{\underbrace{\mathsf{E}[e^{\lambda X}]}_{N(0,1)\text{ mgf}} - 1\big\} = \frac{e^{-\lambda}}{\lambda}\big\{e^{\lambda^2/2} - 1\big\}.$$

   Alternative Solution.

$$\begin{aligned}\mathsf{E}\left[\int_0^X t^N\,dt\right] &= \mathsf{E}\left[\frac{X^{N+1}}{N+1}\right] = \sum_{n=0}^{\infty} \mathsf{E}\left[\frac{X^{N+1}}{N+1}\middle| N=n\right]\mathsf{P}(N=n) = \sum_{n=0}^{\infty} \mathsf{E}\left[\frac{X^{n+1}}{n+1}\middle| N=n\right]\mathsf{P}(N=n) \\ &= \sum_{n=0}^{\infty} \mathsf{E}[X^{n+1}]\frac{\mathsf{P}(N=n)}{n+1} = \sum_{k=1}^{\infty} \mathsf{E}[X^{2k}]\frac{\mathsf{P}(N=2k-1)}{2k} = \sum_{k=1}^{\infty} \frac{(2k)!}{2^k k!} \cdot \frac{\mathsf{P}(N=2k-1)}{2k} \\ &= \sum_{k=1}^{\infty} \frac{(2k)!}{2^k k!} \cdot \frac{1}{2k} \cdot \frac{\lambda^{2k-1}}{(2k-1)!} e^{-\lambda} = \frac{e^{-\lambda}}{\lambda} \sum_{k=1}^{\infty} \frac{(\lambda^2/2)^k}{k!} = \frac{e^{-\lambda}}{\lambda}\big[e^{\lambda^2/2} - 1\big].\end{aligned}$$

3. Let $X_0, X_1, \dots, X_n$ be random variables applied as input to the moving-average filter with coefficients $h_0, \dots, h_n$. The system output is

$$Y_i := \sum_{k=0}^{i} h_k X_{i-k}, \quad i = 0, \dots, n.$$

Assume $Y_0, \dots, Y_n$ are jointly Gaussian. Find conditions on the filter coefficients that force $X_0, \dots, X_n$ to be jointly Gaussian. **Justify your answer.**

Solution. Observe that

$$\begin{aligned} Y_0 &= h_0 X_0 \\ Y_1 &= h_0 X_1 + h_1 X_0 \\ Y_2 &= h_0 X_2 + h_1 X_1 + h_2 X_0 \\ &\vdots \\ Y_n &= h_0 X_n + h_1 X_{n-1} + \cdots + h_{n-1} X_1 + h_n X_0, \end{aligned} \tag{1}$$

or in matrix form

$$\begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 \\ & & & \ddots & \\ h_n & h_{n-1} & \cdots & & h_0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}.$$

Denoting the left-hand side by Y and the right-hand side by HX, if we can write $X = H^{-1}Y$, then X will be Gaussian. Since H is lower triangular, its determinant is the product of the diagonal elements; hence, $\det H = h_0^{n+1}$, which is nonzero if and only if $h_0 \neq 0$. So, if $h_0 \neq 0$, then $X_0, \dots, X_n$ must be jointly Gaussian.

Alternative Solution. From the first equation in (1), we have $X_0 = Y_0/h_0$, assuming $h_0 \neq 0$. Using this result in the second equation in (1) yields

$$X_1 = Y_1/h_0 - h_1 X_0/h_0 = Y_1/h_0 - h_1 Y_0/h_0^2,$$

which expresses X_1 as a linear combination of Y_0 and Y_1. Continuing in this way, we can express X_i as a linear combination of $Y_0, \dots, Y_i$. Hence, we can explicitly solve for $Y = H^{-1}X$, and it follows that X is Gaussian if $h_0 \neq 0$.

4. Let X be a random vector with zero-mean and nonsingular covariance matrix C_X. Put $Y := MX$, where M is a given full-rank $m \times n$ matrix with $m \leq n$. Find the *linear* MMSE estimator of X based on Y. Express your answer in terms of M, C_X, and Y.

   Solution. First note that since X is zero mean, so is $Y := MX$. Next, $C_Y = MC_XM'$, and $C_{XY} = C_XM'$. We must solve $AC_Y = C_{XY}$ or $A(MC_XM') = C_XM'$. Hence, $A = C_XM'(MC_XM')^{-1}$, and the linear MMSE estimator of X based on Y is $AY = C_XM'(MC_XM')^{-1}Y$.

   How did we know that MC_XM' is invertible? Since this is a square matrix, it suffices to show that it is nonsingular; i.e., we must show that $MC_XM'y = 0$ implies $y = 0$. An easy way to see this is to write $MC_XM' = (C_X^{1/2}M')'(C_X^{1/2}M')$. If $MC_XM'y = 0$, then $0 = y'[MC_XM'y] = \|C_X^{1/2}M'y\|^2$. Then since C_X is nonsingular, so is $C_X^{1/2}$, which implies $M'y = 0$. Since M has full rank and $m \leq n$, the rows of M (i.e., the columns of M') are linear independent. Therefore, $M'y = 0$ implies $y = 0$.

5. Let X and Z be independent random variables with $X \sim \exp(\lambda)$ and $\mathsf{P}(Z = \pm 1) = 1/2$. Find the density of $Y := ZX$. **For full credit, simplify your answer as much as possible.**

   Solution. Using the law of total probability, substitution, and independence,

$$
\begin{aligned}
F_Y(y) = \mathsf{P}(Y \leq y) &= \mathsf{P}(ZX \leq y) = \mathsf{P}(ZX \leq y|Z = 1)/2 + \mathsf{P}(ZX \leq y|Z = -1)/2 \\
&= \mathsf{P}(X \leq y)/2 + \mathsf{P}(X \geq -y)/2 \\
&= F_X(y)/2 + [1 - F_X(-y)]/2, \qquad \text{since } X \text{ has a continuous cdf.}
\end{aligned}
$$

   Differentiating yields

$$
\begin{aligned}
f_Y(y) = \tfrac{1}{2}[f_X(y) - f_X(-y)(-1)] = \tfrac{1}{2}[f_X(y) + f_X(-y)] &= \tfrac{\lambda}{2}[e^{-\lambda y}u(y) + e^{-\lambda(-y)}u(-y)] \\
&= \begin{cases} \tfrac{\lambda}{2}e^{-\lambda y}, & y > 0, \\ \tfrac{\lambda}{2}e^{\lambda y}, & y < 0, \end{cases} \\
&= \frac{\lambda}{2}e^{-\lambda|y|}, \quad y \neq 0.
\end{aligned}
$$

   Thus, $Y \sim \text{Laplace}(\lambda)$.