ECE 730
Final Exam
18 December 2013
5:05–7:05 pm in 2535 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring two sheets of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

Some trigonometric identities:

$$\begin{aligned}
e^{j\theta} + e^{-j\theta} &= 2\cos\theta \\
e^{j\theta} - e^{-j\theta} &= 2j\sin\theta
\end{aligned}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\begin{aligned}
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 1 - 2\sin^2 A \qquad \Rightarrow (1 - \cos 2A) = 2\sin^2 A \\
&= 2\cos^2 A - 1 \qquad \Rightarrow (1 + \cos 2A) = 2\cos^2 A
\end{aligned}$$

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\sin 2A &= 2\sin A \cos A
\end{aligned}$$

1. Let X_t be white noise with power spectral density $S_X(f) = N_0/2$. Suppose that X_t is applied to an LTI zero-order-hold system with impulse response $h(t) = I_{[0,T]}(t)$, where $T > 0$ is a given hold duration. Denote the system output by Y_t. Express $\int_{-\infty}^{\infty} R_Y(\tau)\,d\tau$ in closed form.

2. Let $\{W_t, t \geq 0\}$ be a standard Wiener process, and let $g \in L^2[0,\infty)$ be given. Define a new random process by
$$X_t := \int_0^t g(\tau)\,dW_\tau, \quad t \geq 0.$$
For $0 \leq s < t$, find the conditional characteristic function of X_t given X_s, i.e., $\mathsf{E}[e^{j\nu X_t}|X_s]$. **Justify your answer.**

3. Let $Y_1, Y_2, \ldots$ be i.i.d. with common density f, and let g be another density. For simplicity, assume both densities are strictly positive, and assume that the divergence
$$\mathscr{D}(f\|g) := \int_{-\infty}^{\infty} f(y) \log \frac{f(y)}{g(y)}\,dy$$
is finite. Put
$$w(y) := \frac{f(y)}{g(y)}$$
and
$$X_n := \prod_{k=1}^{n} w(Y_k).$$
Determine whether or not $X_n^{1/n}$ converges almost surely. If it does, explain why and identify the limit; if not, give a counterexample.

4. Suppose X_n converges in distribution to X, and Y_n converges in distribution to Y. Does $X_n + Y_n$ converge in distribution to $X + Y$? **Prove it is true or give a counterexample.**

5. Consider a sequence of random variables X_n such that $\mathsf{E}[X_n] \to 0$. If $X_n \geq 0$, then writing $\mathsf{E}[|X_n - 0|] = \mathsf{E}[|X_n|] = \mathsf{E}[X_n] \to 0$ shows that X_n converges in mean of order one to zero. What if we do *not* have $X_n \geq 0$? Give an example of a sequence having *all* of the following properties:
   - $\mathsf{E}[X_n] > 0$,
   - $\mathsf{E}[X_n] \to 0$,
   - X_n does *not* converge in mean of order one to zero.