ECE 730
Final Exam
18 December 2013
5:05–7:05 pm in 2535 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring two sheets of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

Some trigonometric identities:

$$\begin{aligned}
e^{j\theta} + e^{-j\theta} &= 2\cos\theta \\
e^{j\theta} - e^{-j\theta} &= 2j\sin\theta
\end{aligned}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\begin{aligned}
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 1 - 2\sin^2 A & \Rightarrow (1 - \cos 2A) = 2\sin^2 A \\
&= 2\cos^2 A - 1 & \Rightarrow (1 + \cos 2A) = 2\cos^2 A
\end{aligned}$$

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\sin 2A &= 2\sin A \cos A
\end{aligned}$$

1. Let X_t be white noise with power spectral density $S_X(f) = N_0/2$. Suppose that X_t is applied to an LTI zero-order-hold system with impulse response $h(t) = I_{[0,T]}(t)$, where $T > 0$ is a given hold duration. Denote the system output by Y_t. Express $\int_{-\infty}^{\infty} R_Y(\tau)\,d\tau$ in closed form.

   Solution. Since $S_Y(f) = \int_{-\infty}^{\infty} R_Y(\tau)e^{-j2\pi f\tau}\,d\tau$, we see that $S_Y(0)$ is the desired integral. Since $S_Y(f) = |H(f)|^2 S_X(f)$, and since $S_X(0) = N_0/2$, all we have to find is $H(0) = \int_0^T 1\,dt = T$. Thus, $\int_{-\infty}^{\infty} R_Y(\tau)\,dt = |H(0)|^2 S_X(0) = T^2 N_0/2$.
   Alternative Solution 1. A slightly longer solution requires finding $S_Y(f) = |H(f)|^2 S_X(f)$ for all f, not just $f = 0$. The easiest way to find $|H(f)|$ is to realize that it is equal to the absolute value of the Fourier transform of $h(t+T/2) = I_{[-T/2,T/2]}(t)$. From the table,

   $$|H(f)| = T\left|\frac{\sin(\pi T f)}{\pi T f}\right| = T|\operatorname{sinc}(Tf)|.$$

   Alternatively, by direct calculation,

   $$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi f t}\,dt = \int_0^T e^{-j2\pi f t}\,dt = \frac{e^{-j2\pi f t}}{-j2\pi f}\bigg|_0^T = \frac{1-e^{-j2\pi f T}}{j2\pi f}.$$

   At this point, we can write

   $$\frac{1-e^{-j2\pi f T}}{j2\pi f} = e^{-j\pi f T}\frac{e^{j\pi f T}-e^{-j\pi f T}}{j2\pi f} = e^{-j\pi f T}\,T\,\frac{e^{j\pi f T}-e^{-j\pi f T}}{2j(\pi f T)} = e^{-j\pi f T}\,T\,\frac{\sin(\pi f T)}{\pi f T},$$

   or, since we only need $|H(f)|^2$,

   $$|H(f)|^2 = H(f)H(f)^* = \frac{1-e^{-j2\pi f T}}{j2\pi f}\cdot\frac{1-e^{j2\pi f T}}{-j2\pi f} = \frac{2[1-\cos(2\pi f T)]}{(2\pi f)^2} = \frac{4\sin^2(\pi f T)}{(2\pi f)^2},$$

   which is equal to $T^2\operatorname{sinc}^2(Tf)$. In any case, $S_Y(f) = T^2\operatorname{sinc}^2(Tf)N_0/2$. Taking $f = 0$ again yields $T^2 N_0/2$.
   Alternative Solution 2. Using (10.17), write

   $$\begin{aligned}
   \int_{-\infty}^{\infty} R_Y(\tau)\,d\tau &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\beta)\left(\int_{-\infty}^{\infty} h(\theta)R_X(\tau-\beta+\theta)\,d\theta\right)d\beta\right]d\tau \\
   &= (N_0/2)\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\beta)\left(\int_{-\infty}^{\infty} h(\theta)\delta(\tau-\beta+\theta)\,d\theta\right)d\beta\right]d\tau \\
   &= (N_0/2)\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\beta)h(\beta-\tau)\,d\beta\right]d\tau \\
   &= (N_0/2)\int_{-\infty}^{\infty} h(\beta)\left[\int_{-\infty}^{\infty} h(\beta-\tau)\,d\tau\right]d\beta \\
   &= (N_0/2)\int_{-\infty}^{\infty} h(\beta)\left[\int_{-\infty}^{\infty} h(s)\,ds\right]d\beta = (N_0/2)T^2, \quad \text{from the def. of } h.
   \end{aligned}$$

2. Let $\{W_t, t \geq 0\}$ be a standard Wiener process, and let $g \in L^2[0,\infty)$ be given. Define a new random process by
$$X_t := \int_0^t g(\tau)\,dW_\tau, \quad t \geq 0.$$
For $0 \leq s < t$, find the conditional characteristic function of X_t given X_s, i.e., $\mathsf{E}[e^{j\nu X_t}|X_s]$.
Justify your answer.

Solution. We begin by writing
$$\mathsf{E}[e^{j\nu X_t}|X_s] = \mathsf{E}[e^{j\nu(X_t - X_s + X_s)}|X_s] = \mathsf{E}[e^{j\nu(X_t - X_s)}e^{j\nu X_s}|X_s] = \mathsf{E}[e^{j\nu(X_t - X_s)}|X_s]e^{j\nu X_s}.$$
Hence, it suffices to compute
$$\mathsf{E}[e^{j\nu(X_t - X_s)}|X_s] = \mathsf{E}[e^{j\nu(X_t - X_s)}|X_s - X_0], \quad \text{since } X_0 \equiv 0.$$
Next observe that
$$\mathsf{E}[(X_t - X_s)(X_s - X_0)] = \mathsf{E}\bigg[\bigg(\int_s^t g(\tau)\,dW_\tau\bigg)\bigg(\int_0^s g(\tau)\,dW_\tau\bigg)\bigg] = \int_0^\infty g(\tau)^2 I_{[s,t]}(\tau) I_{[0,s]}(\tau)\,d\tau = 0.$$
Thus, $X_t - X_s$ and $X_s - X_0$ are uncorrelated. Since $\{X_t\}$ is Gaussian by HW Problem 14.17, the increments $X_t - X_s$ and $X_s - X_0$ are independent. Therefore,
$$\mathsf{E}[e^{j\nu(X_t - X_s)}|X_s - X_0] = \mathsf{E}[e^{j\nu(X_t - X_s)}].$$
Since $X_t - X_s$ is Gaussian, we just need to find its mean and variance. Since Wiener integrals have zero mean, $\mathsf{E}[X_t - X_s] = 0$. Also,
$$\mathsf{E}[(X_t - X_s)^2] = \mathsf{E}\bigg[\bigg(\int_s^t g(\tau)dW_\tau\bigg)^2\bigg] = \int_s^t g(\tau)^2\,d\tau.$$
Thus, $X_t - X_s \sim N(0, \int_s^t g(\tau)^2\,d\tau)$, and
$$\mathsf{E}[e^{j\nu X_t}|X_s] = \exp\bigg[j\nu X_s - \frac{\nu^2}{2}\int_s^t g(\tau)^2\,d\tau\bigg]. \qquad (*)$$

Alternative Solution. The key fact we need is that $\{X_t\}$ is a Gaussian process by HW Problem 14.17. Hence, the conditional distribution of X_t given X_s is completely determined by $\mathsf{E}[X_t|X_s]$ and the error covariance $C_{X_t|X_s} := C_{X_t} - AC_{X_sX_t}$, where A solves $AC_{X_s} = C_{X_tX_s}$. Since Wiener integrals have zero mean,
$$C_{X_t} = \mathsf{E}[X_t^2] = \mathsf{E}\bigg[\bigg(\int_0^t g(\tau)dW_\tau\bigg)^2\bigg] = \int_0^t g(\tau)^2\,d\tau, \quad C_{X_s} = \mathsf{E}[X_s^2] = \int_0^s g(\tau)^2\,d\tau,$$
and since we have a scalar process,
$$C_{X_sX_t} = C_{X_tX_s} = \mathsf{E}[X_tX_s] = \mathsf{E}\bigg[\bigg(\int_0^t g(\tau)\,dW_\tau\bigg)\bigg(\int_0^s g(\tau)\,dW_\tau\bigg)\bigg] = \int_0^s g(\tau)^2\,d\tau, \quad \text{since } s < t.$$
It now follows that $A = 1$, $\mathsf{E}[X_t|X_s] = X_s$, and $C_{X_t|X_s} = \int_s^t g(\tau)^2\,d\tau$. In other words, $X_t|X_s \sim N(X_s, C_{X_t|X_s})$ and so $(*)$ holds.

3. Let $Y_1, Y_2, \ldots$ be i.i.d. with common density f, and let g be another density. For simplicity, assume both densities are strictly positive, and assume that the divergence

$$\mathscr{D}(f\|g) := \int_{-\infty}^{\infty} f(y) \log \frac{f(y)}{g(y)}\, dy$$

is finite. Put

$$w(y) := \frac{f(y)}{g(y)}$$

and

$$X_n := \prod_{k=1}^{n} w(Y_k).$$

Determine whether or not $X_n^{1/n}$ converges almost surely. If it does, explain why and identify the limit; if not, give a counterexample.

Solution. Consider the sequence

$$Z_n := \log X_n^{1/n} = \tfrac{1}{n} \log X_n = \frac{1}{n} \sum_{k=1}^{n} \log w(Y_k).$$

Since the Y_k have density f,

$$\mathsf{E}[\log w(Y_k)] = \int_{-\infty}^{\infty} f(y) \log w(y)\, dy = \int_{-\infty}^{\infty} f(y) \log \frac{f(y)}{g(y)}\, dy = \mathscr{D}(f\|g),$$

which is assumed finite. By the strong law of large numbers, $Z_n \stackrel{\text{a.s.}}{\to} \mathscr{D}(f\|g)$. Hence, $X_n^{1/n} = \exp(Z_n) \stackrel{\text{a.s.}}{\to} \exp[\mathscr{D}(f\|g)]$.

4. Suppose X_n converges in distribution to X, and Y_n converges in distribution to Y. Does $X_n + Y_n$ converge in distribution to $X + Y$? **Prove it is true or give a counterexample.**

Discussion. Here are two ways to see how we might construct a counterexample. First suppose X_n and Y_n are independent. Then

$$\mathsf{E}[e^{j\nu(X_n+Y_n)}] = \mathsf{E}[e^{j\nu X_n}]\mathsf{E}[e^{j\nu Y_n}] \to \mathsf{E}[e^{j\nu X}]\mathsf{E}[e^{j\nu Y}].$$

However, there is no requirement that X and Y be independent. Hence, if we can find X and Y with the correct marginal distributions, but with X and Y dependent, we will have a counterexample. A second approach is to make X_n and Y_n dependent, but X and Y independent.

Solution 1. Consider the case in which X_n, Y_n, and X are all $N(0,1)$ with X_n and Y_n being independent for each n. Put $Y := -X$ so that Y is also $N(0,1)$. Then $F_{X_n}(x) = F_X(x)$ for all n and $F_{Y_n}(y) = F_Y(y)$ for all n. However, $X_n + Y_n \sim N(0,2)$, while $X + Y = 0$.

Solution 2. Let X_n, X, and Y be uniform$[0,1]$, with X and Y being independent. Put $Y_n := 1 - X_n$ so that Y_n is also uniform$[0,1]$. Then $X_n + Y_n = 1$, but $X + Y$ has a triangular density on $[0,2]$.

5. Consider a sequence of random variables X_n such that $\mathsf{E}[X_n] \to 0$. If $X_n \geq 0$, then writing $\mathsf{E}[|X_n - 0|] = \mathsf{E}[|X_n|] = \mathsf{E}[X_n] \to 0$ shows that X_n converges in mean of order one to zero. What if we do *not* have $X_n \geq 0$? Give an example of a sequence having *all* of the following properties:
   - $\mathsf{E}[X_n] > 0$,
   - $\mathsf{E}[X_n] \to 0$,
   - X_n does *not* converge in mean of order one to zero.

   Solution. Let X be any random variable satisfying the conditions $0 < \mathsf{E}[|X|] < \infty$ and $\mathsf{E}[X] = 0$. Let m_n be any sequence of positive numbers converging to zero. Put $X_n := m_n + X$. Then $\mathsf{E}[X_n] = m_n$, which is positive and converges to zero. Furthermore,

$$\mathsf{E}[|X_n - X|] = m_n \to 0$$

   shows that X_n converges in mean of order one to X. But the condition $\mathsf{E}[|X|] > 0$ means that X is *not* the zero random variable. Hence, X_n does not converge in mean of order one to zero. For a specific example, use $X \sim N(0,1)$ and $m_n = 1/n$. Here it is obvious that X is not the zero random variable.

   Alternative Solution. Suppose you did not start with a random variable X as above, but instead started with random variables Y_n with densities $f(y - m_n)$, where again m_n is a sequence of positive numbers converging to zero, and f is a density with zero mean. In this case, let X also have density f, and put $X_n := m_n + X$ so that the density of X_n is the same as the density of Y_n. Then by the argument above, $\mathsf{E}[|X_n|] \not\to 0$, and since Y_n and X_n have the same density,

$$\mathsf{E}[|Y_n|] = \mathsf{E}[|X_n|] \not\to 0.$$

   Alternative Solution. Suppose $\mathsf{P}(X_n = 1 + 1/n) = 1/2$ and $\mathsf{P}(X_n = -1) = 1/2$. Then $\mathsf{E}[X_n] = 1/(2n)$, which is positive and converges to zero, while $\mathsf{E}[|X_n|] = 1 + 1/(2n) \to 1 \neq 0$.