ECE 730
Exam 1
27 October 2014
5:15–6:45 pm in 3534 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring one sheet of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

1. Let X and W be independent random variables with zero means and unit variances. If $Y := \beta X + \sigma W$, find the **linear** MMSE estimate of X based on Y. Your answer should be an explicit formula in terms of β, σ, and Y. **Justify your answer.**

   Solution. From Chapter 8, we know that the linear MMSE estimate of X based on Y is equal to $A(Y - m_Y) + m_X$, where $AC_Y = C_{XY}$. In this problem, $m_X := \mathsf{E}[X] = 0$ and $\mathsf{E}[W] = 0$ are given. Hence, $m_Y = \mathsf{E}[\beta X + \sigma W] = 0$. Next, since X and W are independent $\mathsf{var}(\beta X + \sigma W) = \beta^2 \mathsf{var}(X) + \sigma^2 \mathsf{var}(W) = \beta^2 + \sigma^2$. Thus, $C_Y = \beta^2 + \sigma^2$. More easily, $C_{XY} = \mathsf{E}[X(\beta X + \sigma W)] = \beta \mathsf{E}[X^2] = \beta$ since X and W are independent with zero means and unit variances. Putting this all together, we have $A = \beta/(\beta^2 + \sigma^2)$ and the linear MMSE estimate is $\beta Y/(\beta^2 + \sigma^2)$.

2. Let $\Omega := (-\infty, \infty)$, and let $\mathscr{A}$ denote the collection of all subsets of Ω of the form $(a,b]$, $(-\infty, a]$, and (b, ∞) for all a and b with $-\infty < a < b < \infty$. Let $\mathscr{A}$ also include Ω and the empty set $\varnothing$. Determine whether or not $\mathscr{A}$ is a σ-algebra. **Justify your answer.**

   Solution. No. Notice that $(0,1]^{\mathrm{c}} = (-\infty, 0] \cup (1, \infty)$ is the union of sets in $\mathscr{A}$, but is not itself in $\mathscr{A}$.
   Alternative Solution. Consider the sequence $A_n := (1 - 1/n, 1] \in \mathscr{A}$. Then $\bigcap_{n=1}^{\infty} A_n = \{1\} \notin \mathscr{A}$.

3. On Campus Drive, the speed limit is 40 mph, and vehicle speeds have probability density*

$$f(x) := \begin{cases} \frac{1}{5} e^{-(x-35)/5}, & x \geq 35, \\ 0, & x < 35. \end{cases}$$

   A police officer issues a ticket if a vehicle is going faster than 47 mph. If 10 vehicles pass the police officer, find the variance of the number of tickets issued. **State any additional assumptions you need to compute your answer. Justify the steps of your calculation.**

   Solution. Let X_k denote the speed of the kth vehicle. Then the number of tickets written is

$$T := \sum_{k=1}^{10} I_{(47,\infty)}(X_k).$$

   We must compute $\mathsf{var}(T)$. We **assume** the X_k are **i.i.d.**† with density f. Then T is the sum of the i.i.d. Bernoulli(p) random variables $I_{(47,\infty)}(X_k)$. Hence, T is binomial$(10, p)$, where, with $\lambda := 1/5$, $x_0 := 35$, and $x = 47$,

$$p := \mathsf{P}(X_k > x) = \int_x^{\infty} \lambda e^{-\lambda(\xi - x_0)}\, d\xi = -e^{-\lambda(\xi - x_0)} \bigg|_{\xi = x}^{\xi = \infty} = e^{-\lambda(x - x_0)} = e^{-(47-35)/5} = e^{-12/5}.$$

   Since T is binomial, $\mathsf{var}(T) = 10p(1-p) = 10e^{-12/5}(1 - e^{-12/5}) \approx 0.82488$.

*Although it was not required to do the problem, did you notice that the mean value of this density is 40 mph?

†Do you think this is a reasonable assumption? If the cars near you are speeding, you are likely to speed also. On the other hand, if the cars are a minute or two apart, then their speeds may be independent.

4. Let X and Y be independent random variables with $X \sim \text{Erlang}(m=2, \lambda=1)$ and $Y \sim \text{uniform}[0,1/2]$. Simplify
$$\mathsf{E}\bigg[\int_0^Y e^{tX}\,dt\bigg].$$
Your answer should have NO integrals. **Justify the steps of your calculation.**

Solution. To begin, write
$$\mathsf{E}\bigg[\int_0^Y e^{tX}\,dt\bigg] = \mathsf{E}\bigg[\frac{e^{YX}-1}{X}\bigg] = \mathsf{E}\bigg[\frac{e^{YX}}{X}\bigg] - \mathsf{E}[1/X].$$
Then compute $\mathsf{E}[1/X] = \int_0^\infty (1/x) f_X(x)\,dx = \int_0^\infty (1/x)\cdot xe^{-x}\,dx = \int_0^\infty e^{-x}\,dx = 1$. Next, using the law of total probability, substitution, and independence,
$$\mathsf{E}\bigg[\frac{e^{YX}}{X}\bigg] = \int_{-\infty}^\infty \mathsf{E}\bigg[\frac{e^{YX}}{X}\bigg|Y=y\bigg] f_Y(y)\,dy = 2\int_0^{1/2} \mathsf{E}\bigg[\frac{e^{yX}}{X}\bigg|Y=y\bigg]\,dy = 2\int_0^{1/2} \mathsf{E}\bigg[\frac{e^{yX}}{X}\bigg]\,dy.$$
Now compute
$$\mathsf{E}\bigg[\frac{e^{yX}}{X}\bigg] = \int_{-\infty}^\infty \frac{e^{yx}}{x} f_X(x)\,dx = \int_0^\infty \frac{e^{yx}}{x}\cdot xe^{-x}\,dx = \int_0^\infty e^{x(y-1)}\,dx = \frac{e^{x(y-1)}}{y-1}\bigg|_{x=0}^{x=\infty} = \frac{1}{1-y},$$
where the last step uses the fact that $y \in [0,1/2]$ implies $y-1<0$. Putting this all together, we have
$$\mathsf{E}\bigg[\frac{e^{YX}}{X}\bigg] = 2\int_0^{1/2} \frac{1}{1-y}\,dy = -2\ln(1-y)\bigg|_{y=0}^{y=1/2} = 2\ln 1 - 2\ln(1/2) = -2\ln(1/2) = 2\ln 2,$$
and then
$$\mathsf{E}\bigg[\int_0^Y e^{tX}\,dt\bigg] = (2\ln 2) - 1.$$

Alternative Solution. Using the law of total probability, substitution, and independence, write
$$\begin{aligned}
\mathsf{E}\bigg[\int_0^Y e^{tX}\,dt\bigg] &= \int_0^{1/2} \mathsf{E}\bigg[\int_0^Y e^{tX}\,dt\bigg|Y=y\bigg] f_Y(y)\,dy = \int_0^{1/2} \mathsf{E}\bigg[\int_0^y e^{tX}\,dt\bigg|Y=y\bigg] f_Y(y)\,dy \\
&= \int_0^{1/2} \mathsf{E}\bigg[\int_0^y e^{tX}\,dt\bigg] f_Y(y)\,dy = \int_0^{1/2} \bigg(\int_0^y \mathsf{E}[e^{tX}]\,dt\bigg) f_Y(y)\,dy \\
&= \int_0^{1/2} \bigg(\int_0^y \frac{1}{(1-t)^2}\,dt\bigg) f_Y(y)\,dy = \int_0^{1/2} \bigg(\frac{1}{1-t}\bigg|_{t=0}^{t=y}\bigg) f_Y(y)\,dy \\
&= \int_0^{1/2} \bigg[\frac{1}{1-y} - 1\bigg] f_Y(y)\,dy = -2\bigg[\ln(1-y) + y\bigg]\bigg|_{y=0}^{y=1/2} = -1 + 2\ln 2.
\end{aligned}$$

5. Let X and Y be jointly Gaussian random vectors. Let $g(y) := \mathsf{E}[X|Y = y]$. Determine whether or not the error $X - g(Y)$ is a Gaussian random vector. **Justify your answer.**

Solution. From Chapter 9 we know that $g(y) := \mathsf{E}[X|Y = y] = A(y - m_Y) + m_X$ where $AC_Y = C_{XY}$. Observe that

$$X - g(Y) = X - \{A(Y - m_Y) + m_X\} = X - AY + Am_Y - m_X = \begin{bmatrix} I & -A \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + (Am_Y - m_X)$$

is an affine transformation of the Gaussian random vector $[X', Y']'$. Hence, $X - g(Y)$ is a Gaussian random vector (recall the gray box on p. 365).

Remark. Now that you know the error is Gaussian, rewrite it as $(X - m_X) - A(Y - m_Y)$. This clearly has zero mean, and by the calculations used for review problem 8.34, the error has covariance matrix $C_X - AC_{YX}$. Hence, $X - g(Y) \sim N(0, C_X - AC_{YX})$. Note that this is *different* from the *conditional* distribution of X given $Y = y$.