# ECE 730
# Final Exam
# 17 December 2014
# 2:45–4:45 pm in 3534 EH

## 100 Points

**Justify your answers!** **Be precise!**

**Closed Book** **Closed Notes**

**You may bring two sheets of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.**

What does "Justify your answers" mean? It means that when a step in your analysis uses a result you learned in this course, you need to write out what that result is. For example, when you use the law of total probability or the smoothing property, you need to write, "by the law of total prob." or "by the smoothing property" in your exam booklet. You need to let me know that you understand why that step in your analysis is valid. If you don't write it down, I'll assume you don't understand and I will take points off.

1. Let $X_1, X_2, \ldots$ be i.i.d. $N(0,1)$ random variables, and suppose $Y_n := aX_n + bX_{n+1}$ for given constants $a$ and $b$. Find a simple expression for $\mathsf{E}[X_{n+1}|Y_n]$. **Justify the steps of your analysis.**

2. Let $X_t$ be a zero-mean, mean-square-continuous random process for $a \le t \le b$. Let $X_t$ have correlation function $R(t,s)$ and corresponding eigenvalues $\lambda_n$ and eigenfunctions $\varphi_n(t)$ that satisfy

$$\int_a^b R(t,s)\varphi_n(s)\,ds = \lambda_n \varphi_n(t), \quad a \le t \le b.$$

   Express $\mathsf{E}\left[\int_a^b X_t^2\,dt\right]$ in terms of the eigenvalues $\lambda_n$. **Justify the steps of your analysis.**

3. Let $Y_1, Y_2, \ldots$ be i.i.d. with zero mean and finite second moment. Put $X_n := (Y_1 + \cdots + Y_n)^2$. Determine whether or not $X_n$ is a submartingale with respect to $Y_n$. **Justify the steps of your analysis.**

4. Suppose $\frac{|X_n|}{1+|X_n|}$ converges in distribution to 0. Determine whether or not $X_n$ converges in probability to 0. **Justify your answer.**

5. Let $Y$ and $Z$ be independent random variables, and let $X$ be a bounded random variable. Suppose $\mathsf{E}[X|Y,Z]$ depends only in $Y$, say $\mathsf{E}[X|Y,Z] = \widehat{g}(Y)$. Now let $h$ be any bounded function of $X$. Your friend asks you if $\mathsf{E}[h(X)|Y,Z]$ depends only on $Y$. Construct an example to show your friend why the answer is "No." In other words, you must specify:

   (a) a pmf or density for $Y$

   (b) a pmf or density for $Z$

   (c) a conditional pmf or density for $X$ given $Y,Z$

   such that $\mathsf{E}[X|Y=y,Z=z]$ depends only on $y$. ***Furthermore***, you must specify a bounded function $h(x)$ such that $\mathsf{E}[h(X)|Y=y,Z=z]$ depends on both $y$ and $z$.