ECE 730
Final Exam
17 December 2014
2:45–4:45 pm in 3534 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring two sheets of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

What does "Justify your answers" mean? It means that when a step in your analysis uses a result you learned in this course, you need to write out what that result is. For example, when you use the law of total probability or the smoothing property, you need to write, "by the law of total prob." or "by the smoothing property" in your exam booklet. You need to let me know that you understand why that step in your analysis is valid. If you don't write it down, I'll assume you don't understand and I will take points off.

1. Let $X_1, X_2, \ldots$ be i.i.d. $N(0,1)$ random variables, and suppose $Y_n := aX_n + bX_{n+1}$ for given constants a and b. Find a simple expression for $\mathsf{E}[X_{n+1}|Y_n]$. **Justify the steps of your analysis.**

   Solution. Since the X_n are i.i.d. Gaussian, they are jointly Gaussian by Problem 9.4. Since

$$\begin{bmatrix} Y_n \\ X_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_n \\ X_{n+1} \end{bmatrix}$$

   is a linear transformation of a Gaussian random vector, Y_n and X_{n+1} are jointly Gaussian (p. 365). Hence, $\mathsf{E}[X_{n+1}|Y_n]$ is the *linear* MMSE estimate of X_{n+1} based on Y_n (Section 9.5). From linear estimation theory, we can write $\mathsf{E}[X_{n+1}|Y_n] = AY_n$, where A solves $AC_{Y_n} = C_{X_{n+1}Y_n}$. Observe that

$$C_{Y_n} = \mathsf{E}[Y_n^2] = \mathsf{E}[(aX_n + bX_{n+1})^2] = \mathsf{E}[a^2X_n^2 + 2abX_nX_{n+1} + b^2X_{n+1}^2] = a^2 + b^2,$$

   and $C_{X_{n+1}Y_n} = \mathsf{E}[X_{n+1}(aX_n + bX_{n+1})] = b$. Solving $A(a^2+b^2) = b$ or $A = b/(a^2+b^2)$, we have $\mathsf{E}[X_{n+1}|Y_n] = bY_n/(a^2+b^2)$.

2. Let X_t be a zero-mean, mean-square-continuous random process for $a \le t \le b$. Let X_t have correlation function $R(t,s)$ and corresponding eigenvalues λ_n and eigenfunctions $\varphi_n(t)$ that satisfy

$$\int_a^b R(t,s)\varphi_n(s)\,ds = \lambda_n \varphi_n(t), \quad a \le t \le b.$$

   Express $\mathsf{E}\left[\int_a^b X_t^2\,dt\right]$ in terms of the eigenvalues λ_n. **Justify the steps of your analysis.**

   Solution. By the Karhunen–Loève expansion,

$$X_t = \sum_{n=1}^\infty A_n \varphi_n(t), \quad a \le t \le b.$$

   Hence,

$$\begin{aligned}
\mathsf{E}\left[\int_a^b X_t^2\,dt\right] &= \mathsf{E}\left[\int_a^b \left(\sum_{n=1}^\infty A_n\varphi_n(t)\right)\left(\sum_{k=1}^\infty A_k\varphi_k(t)\right)dt\right] \\
&= \mathsf{E}\left[\sum_{n=1}^\infty\sum_{k=1}^\infty A_nA_k \int_a^b \varphi_n(t)\varphi_k(t)\,dt\right] \\
&= \mathsf{E}\left[\sum_{n=1}^\infty\sum_{k=1}^\infty A_nA_k\right], \quad \text{since the } \varphi_n \text{ are orthonormal,} \\
&= \sum_{n=1}^\infty\sum_{k=1}^\infty \mathsf{E}[A_nA_k] \\
&= \sum_{n=1}^\infty \mathsf{E}[A_n^2], \quad \text{since the } A_n \text{ are uncorrelated,} \\
&= \sum_{n=1}^\infty \lambda_n.
\end{aligned}$$

Alternative Solution. Write

$$\mathsf{E}\bigg[\int_a^b X_t^2\,dt\bigg] = \int_a^b \mathsf{E}[X_t^2]\,dt = \int_a^b R(t,t)\,dt = \int_a^b \sum_{k=1}^\infty \lambda_k \varphi_k(t)\varphi_k(t)\,dt, \quad \text{by Mercer's Theorem,}$$

$$= \sum_{k=1}^\infty \lambda_k \int_a^b |\varphi_k(t)|^2\,dt$$

$$= \sum_{k=1}^\infty \lambda_k, \quad \text{since the } \varphi_k \text{ have unit energy on } [a,b].$$

3. Let $Y_1, Y_2, \ldots$ be i.i.d. with zero mean and finite second moment. Put $X_n := (Y_1 + \cdots + Y_n)^2$. Determine whether or not X_n is a submartingale with respect to Y_n. **Justify the steps of your analysis.**

Solution. Yes. Write

$$\begin{aligned}
\mathsf{E}[X_{n+1}|Y_n,\ldots,Y_1] &= \mathsf{E}[([Y_1+\cdots+Y_n]+Y_{n+1})^2|Y_n,\ldots,Y_1] \\
&= \mathsf{E}[(Y_1+\cdots+Y_n)^2 + 2(Y_1+\cdots+Y_n)Y_{n+1} + Y_{n+1}^2|Y_n,\ldots,Y_1] \\
&= \mathsf{E}[X_n + 2(Y_1+\cdots+Y_n)Y_{n+1} + Y_{n+1}^2|Y_n,\ldots,Y_1] \\
&= X_n + 2(Y_1+\cdots+Y_n)\underbrace{\mathsf{E}[Y_{n+1}|Y_n,\ldots,Y_1]}_{=\mathsf{E}[Y_{n+1}]=0} + \underbrace{\mathsf{E}[Y_{n+1}^2|Y_n,\ldots,Y_1]}_{\geq 0} \\
&\geq X_n,
\end{aligned}$$

where the fourth equality uses Problem 14.56 twice and the independence of Y_{n+1} from $Y_n,\ldots,Y_1$.

4. Suppose $\frac{|X_n|}{1+|X_n|}$ converges in distribution to 0. Determine whether or not X_n converges in probability to 0. **Justify your answer.**

Solution. Yes. Since $|X_n|/(1+|X_n|)$ converges in distribution to 0, it also converges in probability to 0. Hence, for every $\varepsilon > 0$,

$$\mathsf{P}\bigg(\frac{|X_n|}{1+|X_n|} \geq \varepsilon\bigg) \to 0.$$

For $0 < \varepsilon < 1$, this is equivalent to

$$\mathsf{P}\bigg(|X_n| \geq \frac{\varepsilon}{1-\varepsilon}\bigg) \to 0.$$

Given any $\delta > 0$, put $\varepsilon := \delta/(1+\delta) \in (0,1)$ so that $\delta = \varepsilon/(1-\varepsilon)$. Then

$$\mathsf{P}(|X_n| \geq \delta) = \mathsf{P}\bigg(|X_n| \geq \frac{\varepsilon}{1-\varepsilon}\bigg) = \mathsf{P}\bigg(\frac{|X_n|}{1+|X_n|} \geq \varepsilon\bigg) \to 0.$$

Alternative Solution 1. Put $Y_n := |X_n|/(1+|X_n|)$. Since Y_n converges in distribution to 0, Y_n converges in probability to 0. Consider the function $g(y) := y/(1-y)$ for $0 \leq y < 1$. This is a continuous function, and so $g(Y_n)$ converges in probability to $g(0) = 0$. But $g(Y_n) = |X_n|$. So $|X_n|$ converges in probability to 0. It follows that X_n itself converges in probability to 0.

Alternative Solution 2. For $0 \le t \le 1$, put $g(t) := t$, and for $t > 1$, put $g(t) = 1$. Then g is a bounded continuous function. Since $Y_n := |X_n|/(1+|X_n|) \in [0,1]$, we can write $Y_n = g(Y_n)$. Hence,

$$\mathsf{E}\left[\frac{|X_n|}{1+|X_n|}\right] = \mathsf{E}[Y_n] = \mathsf{E}[g(Y_n)] \to \mathsf{E}[g(0)] = \mathsf{E}[0] = 0.$$

By Problem 14.10, X_n converges in probability to zero.

5. Let Y and Z be independent random variables, and let X be a bounded random variable. Suppose $\mathsf{E}[X|Y,Z]$ depends only in Y, say $\mathsf{E}[X|Y,Z] = \widehat{g}(Y)$. Now let h be any bounded function of X. Your friend asks you if $\mathsf{E}[h(X)|Y,Z]$ depends only on Y. Construct an example to show your friend why the answer is "No." In other words, you must specify:

   (a) a pmf or density for Y

   (b) a pmf or density for Z

   (c) a conditional pmf or density for X given Y,Z

   such that $\mathsf{E}[X|Y=y,Z=z]$ depends only on y. ***Furthermore***, you must specify a bounded function $h(x)$ such that $\mathsf{E}[h(X)|Y=y,Z=z]$ depends on both y and z.

   Solution. Let Y and Z be independent uniform$[0,1/2]$ random variables, and take

$$\mathsf{P}(X=j|Y=y,Z=z) = \begin{cases} [1-(y+z)]/2, & j=2, \\ y, & j=1, \\ z, & j=0, \\ [1-(y+z)]/2, & j=-2, \\ 0, & \text{otherwise.} \end{cases}$$

   Then $\mathsf{E}[X|Y,Z] = Y$, but $\mathsf{E}[\mathbf{1}_{\{0,1\}}(X)|Y,Z] = \mathsf{P}\big(X \in \{0,1\}|Y,Z\big) = Y+Z$.

   Alternative Solution. Let U, Y, and Z be independent and put $X := Y + UZ$. Then

$$\mathsf{E}[X|Y,Z] = \mathsf{E}[Y+UZ|Y,Z] = Y + Z\,\mathsf{E}[U|Y,Z] = Y + Z\,\mathsf{E}[U], \quad \text{by independence.}$$

   If $\mathsf{E}[U] = 0$, then $\mathsf{E}[X|Y,Z]$ depends only on Y. If we take $h(x) := x^2$, then

$$\begin{aligned} \mathsf{E}[h(X)|Y,Z] = \mathsf{E}\big[Y^2 + 2YZU + U^2Z^2\big|Y,Z\big] &= Y^2 + 2YZ\,\mathsf{E}[U|Y,Z] + Z^2\,\mathsf{E}[U^2|Y,Z] \\ &= Y^2 + Z^2\,\mathsf{E}[U^2], \quad \text{again assuming } \mathsf{E}[U] = 0. \end{aligned}$$

   In order that all the conditional expectations be defined and to apply Problem 13.56, we need Y, UZ, U, Y^2, YZU, U^2Z^2, and U^2 in L^1. We also need $\mathsf{E}[U^2] > 0$. So, to be specific, we could take U, Y, and Z to be i.i.d. uniform$[-1,1]$. This also satisfies the requirement that X be bounded.