ECE 730
Exam 1
21 October 2015
5:15–6:30 pm in 2540 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring one sheet of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

1. Let $Y \sim N(0,\sigma^2)$, and given $Y = y$, let $X \sim \exp(y^2)$. Find $\mathsf{E}[X^2Y^6]$. **Evaluate all integrals.**

   Solution. Write

$$\begin{aligned}\mathsf{E}[X^2Y^2] = \int_{-\infty}^{\infty} \mathsf{E}[X^2Y^6|Y=y]f_Y(y)\,dy &= \int_{-\infty}^{\infty} \mathsf{E}[X^2y^6|Y=y]f_Y(y)\,dy \\ &= \int_{-\infty}^{\infty} \mathsf{E}[X^2|Y=y]y^6 f_Y(y)\,dy \\ &= \int_{-\infty}^{\infty} (2/y^4)y^6 f_Y(y)\,dy = 2\mathsf{E}[Y^2] = 2\sigma^2.\end{aligned}$$

2. There are n students in a classroom, and each student has a random number of pencils X_i, where the X_i are i.i.d. uniformly distributed on $\{0,1,2,3,4,5,6,7,8,9\}$. For fixed k in the range $0,\ldots,n$, find the probability that exactly k students have three or more pencils.

   Solution. Notice that $\mathbf{1}_{\{X_i \ge 3\}} = 1$ if and only if the ith student has three or more students. The number of students having three or more pencils is

$$Y = \sum_{i=1}^{n} \mathbf{1}_{\{X_i \ge 3\}},$$

   and we must find $\mathsf{P}(Y = k)$. Since $Y \sim \text{binomial}(n, p)$, where

$$p = \mathsf{P}(X_i \ge 3) = \mathsf{P}(X_i \in \{3,4,5,6,7,8,9\}) = 7/10 = 0.7,$$

   we have $\mathsf{P}(Y = k) = \binom{n}{k}(0.7)^k(0.3)^{n-k}$.

3. Let U and Y be zero-mean random vectors having given covariance matrices C_Y and C_{UY}. Let A solve $AC_Y = C_{UY}$, where C_Y is *not* assumed to be invertible. Find the *linear* MMSE estimate of $X := [U'\ Y']'$ based on Y. **Justify your answer.**

   Solution. We need to find B such that

$$\begin{aligned}0 &= \mathsf{E}\left[\left(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} Y\right)' \left(\begin{bmatrix} U \\ Y \end{bmatrix} - \begin{bmatrix} A \\ B \end{bmatrix} Y\right)\right] \\ &= \mathsf{E}\left[\big[\,(C_1Y)'\ (C_2Y)'\,\big] \begin{bmatrix} U - AY \\ Y - BY \end{bmatrix}\right] \\ &= \mathsf{E}[(C_1Y)'(U - AY)] + \mathsf{E}[(C_2Y)'(Y - BY)]\end{aligned}$$

   for all C_1 and C_2. In this last line, the first expectation is zero because $AC_Y = C_{UY}$ implies that A satisfies the orthogonality condition for estimating U based on Y. The second expectation will be zero if we take $B = I$. Hence, the optimal estimate is

$$\begin{bmatrix} A \\ I \end{bmatrix} Y.$$

   Alternative Solution. We know that $AC_Y = C_{UY}$. We need to solve

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} C_Y = C_{[U',Y']'Y} = \begin{bmatrix} C_{UY} \\ C_Y \end{bmatrix}.$$

   Taking $A_1 = A$ and $A_2 = I$ does this.

4. Consider random variables $U = X+Y$ and $V = X-Y$. If U and V are jointly Gaussian, determine whether or not X and Y are jointly Gaussian. **Justify your answer.**

   Solution. Yes. Notice that
$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}.$$
   Since $[X,Y]'$ is the linear transformation of the Gaussian random vector $[U,V]'$, $[X,Y]'$ is Gaussian.

5. Let Ω be a nonempty set, and let $\mathscr{A}$ be a σ-algebra of subsets of Ω (but not the collection of *all* subsets of Ω). Fix any set $B \subset \Omega$, where $B \notin \mathscr{A}$. Put $\mathscr{C} := \{A \cap B : A \in \mathscr{A}\}$. Determine whether or not $\mathscr{C}$ is a σ-algebra of B. *Hint:* To address this question, it is essential to take complements of subsets of B relative to B. In other words, if $D \subset B$, then the complement of D relative to B is $D^{\mathrm{c}} \cap B$.

   Solution. Yes. Clearly, $\varnothing$ can be written as $\varnothing \cap B$ with $\varnothing \in \mathscr{A}$. Next, if $C = A \cap B$ with $A \in \mathscr{A}$, then the complement of C relative to B is
$$C^{\mathrm{c}} \cap B = (A \cap B)^{\mathrm{c}} \cap B = (A^{\mathrm{c}} \cup B^{\mathrm{c}}) \cap B = (A^{\mathrm{c}} \cap B) \cup (B^{\mathrm{c}} \cap B) = (A^{\mathrm{c}} \cap B) \cup \varnothing = A^{\mathrm{c}} \cap B,$$
   where, of course, $A^{\mathrm{c}} \in \mathscr{A}$. Finally, if $C_n \in \mathscr{C}$ with $C_n = A_n \cap B$ for $A_n \in \mathscr{A}$, then
$$\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} (A_n \cap B) = \left(\bigcup_{n=1}^{\infty} A_n \right) \cap B,$$
   where this last union belongs to $\mathscr{A}$.