ECE 730
Final Exam
22 December 2015
7:45–9:45 am in 3534 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring two sheets of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

What does "Justify your answers" mean? It means that when a step in your analysis uses a result you learned in this course, you need to write out what that result is. For example, when you use the law of total probability or the smoothing property, you need to write, "by the law of total prob." or "by the smoothing property" in your exam booklet. You need to let me know that you understand why that step in your analysis is valid. If you don't write it down, I'll assume you don't understand and I will take points off.

1. A zero-mean, wide-sense stationary process X_t with correlation function $R_X(\tau) = 1/(1+\tau^2)$ is applied to a linear, time-invariant system with impulse response $h(t) = 6\sin(2\pi t)/(2\pi t)$. Let Y_t denote the response of this system to the input X_t. Find a closed-form expression for $\mathsf{E}[Y_t^2]$.

2. Let W_t be a Wiener process, and suppose $\int_0^\infty g(\tau)^2\,d\tau < \infty$. For $0 < s < t$, evaluate

$$\mathsf{E}\left[(W_t - W_s)\int_0^s g(\tau)\,dW_\tau\right].$$

   Justify the steps of your calculation.

3. We know from Example 13.11 that if $\mathsf{E}[|X_n - X|] \to 0$, then $\mathsf{E}[X_n] \to \mathsf{E}[X]$. Is the converse true? In other words, if $\mathsf{E}[X_n] \to \mathsf{E}[X]$, does it follow that $\mathsf{E}[|X_n - X|] \to 0$? If "yes," give a proof; if "no," give a counterexample.

4. Construct a sequence of random variables X_n converging almost surely to zero, but having

$$\sum_{n=1}^{\infty} \mathsf{P}(|X_n| \geq \varepsilon) = \infty$$

   for some $\varepsilon > 0$.

5. Let $X_0, W_1, W_2, W_3, \ldots$ be independent, L^1 random variables, and let $\varphi(x)$ be a bounded function. Put

$$X_n := X_{n-1} + \varphi(X_{n-1})W_n, \quad n = 1, 2, \ldots.$$

   For the case $n = 2$, derive the formula

$$\mathsf{E}[X_n | X_0, W_1, \ldots, W_{n-1}] = X_{n-1} + \varphi(X_{n-1})\mathsf{E}[W_n].$$

   Justify the steps of your derivation!