ECE 730
Final Exam
22 December 2015
7:45–9:45 am in 3534 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes**

You may bring two sheets of 8.5 in. $\times$ 11 in. paper on which you have prepared formulas.

What does "Justify your answers" mean? It means that when a step in your analysis uses a result you learned in this course, you need to write out what that result is. For example, when you use the law of total probability or the smoothing property, you need to write, "by the law of total prob." or "by the smoothing property" in your exam booklet. You need to let me know that you understand why that step in your analysis is valid. If you don't write it down, I'll assume you don't understand and I will take points off.

1. A zero-mean, wide-sense stationary process X_t with correlation function $R_X(\tau) = 1/(1+\tau^2)$ is applied to a linear, time-invariant system with impulse response $h(t) = 6\sin(2\pi t)/(2\pi t)$. Let Y_t denote the response of this system to the input X_t. Find a closed-form expression for $\mathsf{E}[Y_t^2]$.

   Solution. By (10.24), (10.23), and the tables,

$$\begin{aligned}\mathsf{E}[Y_t^2] = \int_{-\infty}^{\infty} S_Y(f)\,df = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\,df &= \int_{-\infty}^{\infty} |3\mathbf{1}_{[-1,1]}(f)|^2\,\pi e^{-2\pi|f|}\,df \\ &= 9\int_{-1}^{1} \pi e^{-2\pi|f|}\,df \\ &= 18\pi\int_0^1 e^{-2\pi f}\,df = 9(1-e^{-2\pi}).\end{aligned}$$

2. Let W_t be a Wiener process, and suppose $\int_0^\infty g(\tau)^2\,d\tau < \infty$. For $0 < s < t$, evaluate

$$\mathsf{E}\left[(W_t - W_s)\int_0^s g(\tau)\,dW_\tau\right].$$

   Justify the steps of your calculation.

   Solution. Write

$$\begin{aligned}\mathsf{E}\left[(W_t - W_s)\int_0^s g(\tau)\,dW_\tau\right] &= \mathsf{E}\left[\left(\int_0^\infty \mathbf{1}_{[s,t]}(\tau)\,dW_\tau\right)\left(\int_0^\infty g(\tau)\mathbf{1}_{[0,s]}(\tau)\,dW_\tau\right)\right] \\ &= \sigma^2\int_0^\infty \mathbf{1}_{[s,t]}(\tau)\cdot g(\tau)\mathbf{1}_{[0,s]}(\tau)\,d\tau = \sigma^2\int_0^\infty 0\,d\tau = 0.\end{aligned}$$

3. We know from Example 13.11 that if $\mathsf{E}[|X_n - X|] \to 0$, then $\mathsf{E}[X_n] \to \mathsf{E}[X]$. Is the converse true? In other words, if $\mathsf{E}[X_n] \to \mathsf{E}[X]$, does it follow that $\mathsf{E}[|X_n - X|] \to 0$? If "yes," give a proof; if "no," give a counterexample.

   Solution. No. Let Y be any random variable with $0 < \mathsf{E}[|Y|] < \infty$ and $\mathsf{E}[Y] = 0$. Take $X_n = Y$ and $X = 0$. Then $\mathsf{E}[X_n] = \mathsf{E}[Y] = 0 = \mathsf{E}[X]$, but $\mathsf{E}[|X_n - X|] = \mathsf{E}[|Y|] > 0$. Some specific examples include $\mathsf{P}(Y = \pm 1) = 1/2$; $Y \sim N(0,\sigma^2)$; and $Y \sim \text{Laplace}(\lambda)$.

   Alternative Solution 1. We modify the foregoing by allowing X to be any zero-mean random variable whose distribution is *different* from that of Y. Now it is a little harder to argue that $\mathsf{E}[|X_n - X|] \not\to 0$. Suppose otherwise. Then since convergence in mean of order one implies convergence in distribution, $Y = X_n$ converges in distribution to X; i.e., Y and X have the *same* distribution, which is a contradiction.

   Alternative Solution 2. For another, easier modification, take $X = -Y$. In this case, $\mathsf{E}[|X_n - X|] = 2\mathsf{E}[|Y|] > 0$.

4. Construct a sequence of random variables X_n converging almost surely to zero, but having

$$\sum_{n=1}^{\infty} \mathsf{P}(|X_n| \geq \varepsilon) = \infty$$

for some $\varepsilon > 0$.

Solution. Let $U \sim \text{uniform}[0,1]$, and put $X_n := \mathbf{1}_{[0,1/n]}(U)$. We have to demonstrate two things. First, we have to show that $X_n \to 0$ amost surely. Second, we have to show that for some $\varepsilon > 0$, we have $\sum_{n=1}^{\infty} \mathsf{P}(|X_n| \geq \varepsilon) = \infty$.

1) If $U(\omega) > 0$, then $X_n(\omega) = 0$ for $n > 1/U(\omega)$, which implies that $X_n(\omega) \to 0$. Equivalently, $\{U > 0\} \subset \{X_n \to 0\}$, from which it follows that $\mathsf{P}(X_n \to 0) \geq \mathsf{P}(U > 0) = 1$. Hence, $X_n \to 0$ with probability one (i.e., almost surely).

2) Next, for every $0 < \varepsilon < 1$ (but not $\varepsilon \geq 1$), we also have $\mathsf{P}(|X_n| \geq \varepsilon) = \mathsf{P}(U \leq 1/n) = 1/n$, and so $\sum_{n=1}^{\infty} \mathsf{P}(|X_n| \geq \varepsilon) = \sum_{n=1}^{\infty} 1/n = \infty$.

An approach that does <u>not</u> work (but gives a proof of the SLLN under stronger assumptions than Example 14.15). Let $Y_1, Y_2, \ldots$ be i.i.d. with zero mean, even density, and moment generating function $M(s)$ that is finite in a neighborhood of the origin. Put $X_n := (1/n)\sum_{k=1}^{n} Y_k$. By the strong law of large numbers, X_n converges almost surely to zero. However, as we now show, $\sum_{n=1}^{\infty} \mathsf{P}(|X_n| \geq \varepsilon) < \infty$.

Fix any n, and let $s > 0$ be small enough that $M(s/n)$ is finite. Then by the Chernoff bound technique,

$$\begin{aligned}
\mathsf{P}(|X_n| \geq \varepsilon) = 2\mathsf{P}(X_n \geq \varepsilon) = 2\mathsf{P}(sX_n \geq s\varepsilon) &\leq 2e^{-s\varepsilon}\mathsf{E}[e^{sX_n}], \quad \text{by the Markov ineq.} \\
&= 2e^{-s\varepsilon}M(s/n)^n \\
&= 2\exp\bigl[-s\varepsilon + n\ln M(s/n)\bigr] \\
&= 2\exp\bigl[-n\bigl\{(s/n)\varepsilon - \ln M(s/n)\bigr\}\bigr].
\end{aligned}$$

To minimize the right-hand side, we need to maximize the quantity in braces. Although the maximizing value of s may depend on n, the maximum value of the quantity in braces does *not* depend on n. This value is

$$\Psi(\varepsilon) := \sup_{\theta}[\theta\varepsilon - \ln M(\theta)],$$

where the sup is over $\theta \geq 0$ for which $M(\theta)$ is finite. (Taking $\theta = 0$ on the right of the above display shows that $\Psi(\varepsilon) \geq 0$.) Then

$$\Psi(\varepsilon) \geq \varepsilon(s/n) - \ln M(s/n),$$

and we can write

$$\mathsf{P}(|X_n| \geq \varepsilon) \leq 2e^{-n\Psi(\varepsilon)}.$$

Hence, $\sum_{n=1}^{\infty} \mathsf{P}(|X_n| \geq \varepsilon) < \infty$.

5. Let $X_0, W_1, W_2, W_3, \ldots$ be independent, L^1 random variables, and let $\varphi(x)$ be a bounded function. Put

$$X_n := X_{n-1} + \varphi(X_{n-1})W_n, \quad n = 1, 2, \ldots.$$

For the case $n = 2$, derive the formula

$$\mathsf{E}[X_n | X_0, W_1, \ldots, W_{n-1}] = X_{n-1} + \varphi(X_{n-1})\mathsf{E}[W_n].$$

Justify the steps of your derivation!

Solution. To begin, write

$$X_1 = X_0 + \varphi(X_0)W_1, \qquad (*)$$

and note that since $X_0, W_1 \in L^1$ and since φ is bounded, $X_1 \in L^1$. It then similarly follows that $X_2 = X_1 + \varphi(X_1)W_2 \in L^1$. We are now able to write

$$\mathsf{E}[X_2|X_0, W_1] = \mathsf{E}[X_1|X_0, W_1] + \mathsf{E}[\varphi(X_1)W_2|X_0, W_1], \quad \textbf{by Example 13.22 (linearity)}.$$

Eq. $(*)$ tells us that X_1 is a function of X_0 and W_1. Hence, by the **Remark in Problem 13.56**,

$$\mathsf{E}[X_1|X_0, W_1] = X_1.$$

Since $\varphi(X_1)$ is also a function of X_0 and W_1, **Problem 13.56** itself yields

$$\mathsf{E}[\varphi(X_1)W_2|X_0, W_1] = \varphi(X_1)\mathsf{E}[W_2|X_0, W_1].$$

Finally, since W_2 is independent of X_0 and W_1, we have by **Example 13.21** that $\mathsf{E}[W_2|X_0, W_1] = \mathsf{E}[W_2]$. Putting this all together, we have the desired formula,

$$\mathsf{E}[X_2|X_0, W_1] = X_1 + \varphi(X_1)\mathsf{E}[W_2].$$