ECE 730, Lec. 1
Exam 1
Wednesday, 17 Oct. 2018
5:00 pm – 6:30 pm
2534 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes** **No Calculators**

You may bring one sheet of 8.5 × 11 paper with notes written on both sides.

The solutions of a quadratic equation $as^2 + bs + c = 0$ are

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. [20 pts.] Let X and Y be random vectors, and suppose you are given

$$m_X := \mathsf{E}[X] \qquad R_X := \mathsf{E}[XX'] \qquad \text{and} \qquad R_{XY} := \mathsf{E}[XY'].$$
$$m_Y := \mathsf{E}[Y] \qquad R_Y := \mathsf{E}[YY']$$

Express the **linear** MMSE estimate of X based on Y using any of the above quantities as appropriate. If you need the inverse of a matrix, assume it exists.

Solution. We know that the required estimate is $A(Y - m_Y) + m_X$, where A solves $AC_Y = C_{XY}$. Since

$$C_Y = R_Y - m_Y m_Y', \quad \text{and} \quad C_{XY} = R_{XY} - m_X m_Y',$$

we have

$$A = (R_{XY} - m_X m_Y')(R_Y - m_Y m_Y')^{-1}.$$

So the estimate is

$$(R_{XY} - m_X m_Y')(R_Y - m_Y m_Y')^{-1}(Y - m_Y) + m_X.$$

2. [20 pts.] Let $X \sim \exp(\lambda)$, and given $X = x$, let the conditional density of Y be $N(x^3, \sigma^2)$. Compute $\mathsf{E}[Y^2]$. **Show your work**.

Solution. $\mathsf{E}[Y^2] = \mathsf{E}\big[\mathsf{E}[Y^2|X]\big] = \mathsf{E}\big[\sigma^2 + (X^3)^2\big] = \sigma^2 + 6!/\lambda^6$.

3. [20 pts.] Let $Y = (X+W)^2$, where $X \sim N(0, 0.9)$, $W \sim N(0, 0.1)$, and X and W are independent. Find the density of Y. **Justify your answer.**

Solution. Since X and W are independent Gaussians, $X + W$ is also Gaussian, with zero mean and variance $0.9 + 0.1 = 1$, by Problem 4.55(a). Hence, Y is chi-squared with one degree of freedom by Problem 4.46.

4. [20 pts.] If $X \sim \exp(\lambda)$ and $Y \sim \exp(\mu)$ are independent, find the density of $Z := \max(X, Y)$. **Show your work.**

Solution. Since $Z \geq 0$, it suffices to consider $z > 0$. For such z,

$$F_Z(z) = \mathsf{P}(Z \leq z) = \mathsf{P}(\max(X,Y) \leq z) = \mathsf{P}(X \leq z, Y \leq z) = F_X(z)F_Y(z).$$

Then

$$\begin{aligned} f_Z(z) = \frac{d}{dz}F_Z(z) = f_X(z)F_Y(z) + F_X(z)f_Y(z) &= \lambda e^{-\lambda z}(1 - e^{-\mu z}) + (1 - e^{-\lambda z})\mu e^{-\mu z} \\ &= \lambda e^{-\lambda z} + \mu e^{-\mu z} - (\lambda + \mu)e^{-(\lambda+\mu)z}. \end{aligned}$$

5. [20 pts.] Suppose X is a two-dimensional random vector with zero mean and covariance matrix

$$C_X = \begin{bmatrix} \varepsilon & 1 \\ 1 & \varepsilon \end{bmatrix},$$

where $\varepsilon > 1$ is a parameter. Let $X = PY$ be the Karhunen–Loève expansion of X. Find the covariance matrix of Y. Express your answer in terms of ε. *Hint:* This does not require finding the transformation P.

Solution. Since the covariance matrix of Y is diagonal with diagonal elements being the eigenvalues of C_X, it suffices to solve the characteristic equation

$$\det(sI - C_X) = \det \begin{bmatrix} s-\varepsilon & -1 \\ -1 & s-\varepsilon \end{bmatrix} = s^2 - 2\varepsilon s + \varepsilon^2 - 1 = 0. \qquad (*)$$

By the quadratic formula,

$$s = \frac{2\varepsilon \pm \sqrt{4\varepsilon^2 - 4(\varepsilon^2 - 1)}}{2} = \varepsilon \pm 1,$$

and follows that

$$C_Y = \begin{bmatrix} \varepsilon + 1 & 0 \\ 0 & \varepsilon - 1 \end{bmatrix}. \qquad (**)$$

Rather than using the quadratic formula, we could have rearranged the last equation in $(*)$ as $(s-\varepsilon)^2 = 1$. Taking square roots yields $|s - \varepsilon| = 1$, or $s - \varepsilon = \pm 1$, which says that $s = \varepsilon \pm 1$.

Alternative Solution. The eigenvalues of C_X solve

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & \varepsilon \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix},$$

where $[u, v]^\mathsf{T}$ is not the zero vector. From the second equation, $u = (\lambda - \varepsilon)v$. Note that this implies $v \neq 0$, since otherwise $[u, v]^\mathsf{T}$ is the zero vector. Now rewrite the first equation as

$$\begin{aligned} v = (\lambda - \varepsilon)u &= (\lambda - \varepsilon)(\lambda - \varepsilon)v \\ &= (\lambda - \varepsilon)^2 v. \end{aligned}$$

Rewriting this as $[(\lambda - \varepsilon)^2 - 1]v = 0$, and recalling that $v \neq 0$, we must have $(\lambda - \varepsilon)^2 = 1$ as in the above solution. So we again have that C_Y is given by $(**)$.