ECE 730, Lec. 1
Final Exam
Tuesday, 18 Dec. 2018
2:45 pm – 4:45 pm
3444 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes** **No Calculators**

You may bring two sheets of 8.5 $\times$ 11 paper with notes written on both sides.

1. [15 pts.] Consider a linear, time-invariant system with impulse response $h(t) = e^{-t/\tau}u(t)$, where the time constant τ can lie in the range $0 < a \leq \tau \leq b < \infty$. Let Y_t denote the system response to a zero-mean, wide-sense stationary, white-noise process X_t with power spectral density $S_X(f) = N_0/2$. Determine the value of the time constant τ that minimizes the expected instantaneous output power $\mathsf{E}[Y_t^2]$. **Justify your answer.**

   Solution. First recall that $\mathsf{E}[Y_t^2] = R_Y(0)$, and R_Y is the inverse Fourier transform of

$$S_Y(f) = |H(f)|^2 S_X(f) = \frac{N_0/2}{(1/\tau)^2 + (2\pi f)^2}.$$

   By the Fourier Transform table,

$$R_Y(t) = (N_0/2)(\tau/2)e^{-|t|/\tau}.$$

   Since $R_Y(0) = (N_0/2)(\tau/2)$, the minimizing value of τ is $\tau = a$.
   Alternate Solution. Since $R_Y(0) = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\,df = \int_{-\infty}^{\infty} |H(f)|^2 (N_0/2)\,df$, we have by Parseval's Theorem that

$$R_Y(0) = (N_0/2)\int_{-\infty}^{\infty} |h(t)|^2\,dt = (N_0/2)\int_0^{\infty} e^{-t/(\tau/2)}\,dt = (N_0/2)(\tau/2)\int_0^{\infty} \frac{1}{\tau/2} e^{-t/(\tau/2)}\,dt.$$

   This last integrand is the $\exp(2/\tau)$ density, and therefore integrates to one. Hence, $R_Y(0) = (N_0/2)(\tau/2)$ as before.

2. [20 pts.] Let X_n be a time-homogeneous Markov chain with transition probabilities p_{ij} and initial distribution $\nu_i := \mathsf{P}(X_0 = i)$. Determine whether or not

$$\mathsf{P}(X_0 = i | X_1 = j, X_2 = k, X_3 = \ell) = \mathsf{P}(X_0 = i | X_1 = j).$$

   Justify your answer.

   Solution. Yes. It suffices to compute the left-hand side and show that it does not depend on k or ℓ. To begin, write

$$\begin{aligned} \mathsf{P}(X_0 = i, X_1 = j, X_2 = k, X_3 = \ell) &= \mathsf{P}(X_3 = \ell | X_2 = k, X_1 = j, X_0 = i)\mathsf{P}(X_2 = k | X_1 = j, X_0 = i) \\ &\quad \cdot \mathsf{P}(X_1 = j | X_0 = i)\mathsf{P}(X_0 = i) \\ &= \nu_i p_{ij} p_{jk} p_{k\ell}, \qquad \text{by the Markov property.} \end{aligned}$$

   From this expression, we can obtain $\mathsf{P}(X_1 = j, X_2 = k, X_3 = \ell)$ by summing over all possible values that X_0 can take; i.e.,

$$\begin{aligned} \mathsf{P}(X_1 = j, X_2 = k, X_3 = \ell) &= \sum_a \mathsf{P}(X_0 = a, X_1 = j, X_2 = k, X_3 = \ell) \\ &= \sum_a \nu_a p_{aj} p_{jk} p_{k\ell}. \end{aligned}$$

We then compute

$$\frac{\mathsf{P}(X_0=i,X_1=j,X_2=k,X_3=\ell)}{\mathsf{P}(X_1=j,X_2=k,X_3=\ell)}=\frac{\nu_i p_{ij}p_{jk}p_{k\ell}}{\sum_a \nu_a p_{aj}p_{jk}p_{k\ell}}=\frac{\nu_i p_{ij}}{\sum_a \nu_a p_{aj}},$$

which does not depend on k or ℓ as required.

Alternate Solution. Write

$$\begin{aligned}\mathsf{P}(X_0=i|X_1=j,X_2=k,X_3=\ell)&=\frac{\mathsf{P}(X_0=i,X_1=j,X_2=k,X_3=\ell)}{\mathsf{P}(X_1=j,X_2=k,X_3=\ell)}\\ &=\frac{\mathsf{P}(X_3=\ell|X_2=k,X_1=j,X_0=i)\mathsf{P}(X_2=k|X_1=j,X_0=i)\mathsf{P}(X_1=j|X_0=i)\mathsf{P}(X_0=i)}{\mathsf{P}(X_3=\ell|X_2=k,X_1=j)\mathsf{P}(X_2=k|X_1=j)\mathsf{P}(X_1=j)}\\ &=\frac{p_{k\ell}p_{jk}p_{ij}\nu_i}{p_{kl}p_{jk}\mathsf{P}(X_1=j)}\\ &=\frac{p_{ij}\nu_i}{\mathsf{P}(X_1=j)},\end{aligned}$$

which does not depend on k or ℓ as required. Note that to obtain the 3rd line of the display, in the denominator, we need a more general version of the Markov property than the basic one used to obtain the numerator.

3. Suppose $\{N_t, t \geq 0\}$ is a Poisson process, and $\mathsf{E}[N_6] = \mu$.
   (a) [10 pts.] Express $\mathsf{P}(N_2 \leq 2)$ directly in terms of μ.
   (b) [15 pts.] Given $0 < s < t$, find the conditional probability generating function, $\mathsf{E}[z^{N_t}|N_s]$.

Solution.

(a) In general,

$$\mathsf{P}(N_t \leq 2) = e^{-\lambda t}\big(1+\lambda t+\tfrac{1}{2}(\lambda t)^2\big).$$

Since $\mathsf{E}[N_t] = \lambda t$, we see that $\mathsf{E}[N_6] = \lambda \cdot 6 = \mu$, which implies $\lambda = \mu/6$. Specializing the above display to $t = 2$ and $\lambda = \mu/6$, we have, with $\lambda t = \mu/3$,

$$\mathsf{P}(N_2 \leq 2) = e^{-\mu/3}\big(1+\mu/3+\mu^2/18\big).$$

(b) Write

$$\begin{aligned}\mathsf{E}\big[z^{N_t}\big|N_s\big] &= \mathsf{E}\big[z^{N_t-N_s+N_s}\big|N_s\big]\\ &= \mathsf{E}\big[z^{N_t-N_s}z^{N_s}\big|N_s\big]\\ &= \mathsf{E}\big[z^{N_t-N_s}\big|N_s\big]z^{N_s}\\ &= \mathsf{E}\big[z^{N_t-N_s}\big|N_s-N_0\big]z^{N_s}\\ &= \mathsf{E}\big[z^{N_t-N_s}\big]z^{N_s}\\ &= e^{(\mu(t-s)/6)(z-1)}z^{N_s}.\end{aligned}$$

4. Suppose X_n converges in probability to X. Put $Y_n := X_n e^{-X_n}$ and $Y := Xe^{-X}$.
   (a) [10 pts.] Does Y_n converge in probability to Y? **Briefly explain why or why not.**
   (b) [10 pts.] Does $\mathsf{E}[Y_n] \to \mathsf{E}[Y]$? **Briefly explain why or why not.**

Solution.

(a) Yes. Put $h(x) := xe^{-x}$ so that we can write $Y_n = h(X_n)$ and $Y = h(X)$. Since h is continuous $X_n \xrightarrow{\mathsf{P}} X$ implies $h(X_n) \xrightarrow{\mathsf{P}} h(X)$, or $Y_n \xrightarrow{\mathsf{P}} Y$.

(b) No. Since h defined above is not *bounded* and continuous, even though $X_n \xrightarrow{\mathsf{P}} X$ implies $X_n \xrightarrow{\mathsf{D}} X$, we can*not* conclude that $\mathsf{E}[h(X_n)] \to \mathsf{E}[h(X)]$, or equivalently that $\mathsf{E}[Y_n] \to \mathsf{E}[Y]$. To give a counterexample (not required for the exam), consider $X_n := -n\mathbf{1}_{(0,1/n]}(U)$, where $U \sim \text{uniform}(0,1]$. First note that for every ω, $X_n(\omega) \to 0$, which implies $X_n \xrightarrow{\text{a.s.}} 0$, which implies $X_n \xrightarrow{\mathsf{P}} 0$. Next, observe that

$$\mathsf{E}[h(X_n)] = \mathsf{E}[X_n e^{-X_n}] = (-n)e^n \mathsf{P}(U \le 1/n) = -e^n \to -\infty,$$

while with $X \equiv 0$, $\mathsf{E}[h(X)] = \mathsf{E}[Xe^{-X}] = \mathsf{E}[0e^0] = 0$.

5. [20 pts.] Let X_i be a discrete-time, Gaussian random process, and consider the new process,

$$Y_k := \sum_{i=-\infty}^{\infty} h(k-i)X_i,$$

where $h(\cdot)$ is a given, deterministic, finite-duration signal. (Because $h(\cdot)$ is finite duration, for each fixed k, the above sum contains only finitely many terms, which means we do not have to worry about limits.) Determine whether or not Y_k is a Gaussian random process. **Justify your answer.**

Solution. We must show that for arbitrary coefficients $c_1,\ldots,c_n$, the linear combination $\sum_{k=1}^n c_k Y_k$ is a scalar Gaussian random variable. Observe that

$$\sum_{k=1}^n c_k Y_k = \sum_{k=1}^n c_k \left(\sum_{i=-\infty}^{\infty} h(k-i)X_i \right) = \sum_{i=-\infty}^{\infty} \left(\sum_{k=1}^n c_k h(k-i) \right) X_i.$$

Since the inner sum on the right is the convolution of two finite-duration signals, it is finite duration. Hence, the above right-hand side is a finite linear combination of the X_i. Since the X_i are Gaussian, this finite linear combination of the X_i is a scalar Gaussian.

Alternate Solution. Suppose $h(j)$ is nonzero only for $j = 0,\ldots,m$. Then

$$\begin{aligned}
Y_0 &= \sum_{i=-\infty}^{\infty} h(-i)X_i = \sum_{i=-m}^{0} h(-i)X_i \\
Y_1 &= \sum_{i=-\infty}^{\infty} h(1-i)X_i = \sum_{i=-m+1}^{1} h(1-i)X_i \\
&\vdots \\
Y_n &= \sum_{i=-\infty}^{\infty} h(n-i)X_i = \sum_{i=-m+n}^{n} h(n-i)X_i
\end{aligned}$$

We see that $[Y_0,\ldots,Y_n]^\mathsf{T}$ can be written as a matrix times $[X_{-m},\ldots,X_n]^\mathsf{T}$. A similar analysis shows that any finite subsequence of the Y_k forms a Gaussan random vector.