ECE 730, Lec. 1
Exam 1
Monday, 21 Oct. 2019
4:14 pm – 5:45 pm
2540 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes** **No Calculators**

You may bring one sheet of 8.5 $\times$ 11 paper with notes written on both sides.

PART 1 — Straightforward Application of Tools We've Studied

1. [20 pts.] Let $X \sim \text{gamma}(p,1)$, and suppose that given $X = x$, Y is conditionally $\exp(x)$. Also assume that given $X = x$ and $Y = y$, Z is conditionally $N(0,y^2)$. Find $\mathsf{E}[X^4YZ^2]$.

   Solution. Using the law of total probability,

$$
\begin{aligned}
\mathsf{E}[X^4YZ^2] &= \int_0^\infty \int_0^\infty \mathsf{E}[X^4YZ^2|Y=y,X=x]\,f_{XY}(x,y)\,dy\,dx \\
&= \int_0^\infty \int_0^\infty \mathsf{E}[x^4yZ^2|Y=y,X=x]\,f_{XY}(x,y)\,dy\,dx, \quad \text{by substitution,} \\
&= \int_0^\infty \int_0^\infty x^4y\,\underbrace{\mathsf{E}[Z^2|Y=y,X=x]}_{=y^2}\,f_{XY}(x,y)\,dy\,dx, \quad \text{since 2nd moment of } N(0,\sigma^2) \text{ is } \sigma^2, \\
&= \mathsf{E}[X^4Y^3] \\
&= \int_0^\infty \mathsf{E}[X^4Y^3|X=x]\,f_X(x)\,dx, \quad \text{by the law of total probability,} \\
&= \int_0^\infty \mathsf{E}[x^4Y^3|X=x]\,f_X(x)\,dx, \quad \text{by substitution,} \\
&= \int_0^\infty x^4\,\underbrace{\mathsf{E}[Y^3|X=x]}_{=3!/x^3}\,f_X(x)\,dx, \quad \text{since } n\text{th moment of } \exp(\lambda) \text{ is } n!/\lambda^n, \\
&= \int_0^\infty 6x f_X(x)\,dx \\
&= 6\mathsf{E}[X] = 6p, \quad \text{since first moment of gamma}(p,\lambda) \text{ is } \Gamma(1+p)/[\lambda\Gamma(p)] = p/\lambda.
\end{aligned}
$$

2. [20 pts.] A new bridge has 4 cables. Let X_i denote the force on the ith cable. A cable will fail if the force on it exceeds t. If at most one cable fails, the bridge will remain standing.[1]

   (a) Write an expression for the event that the bridge remains standing.

   (b) Assuming the X_i are i.i.d. $\exp(\lambda)$ random variables, find a formula for the probability that the bridge remains standing.

Solution.

(a) The event that the bridge remains standing can be expressed as

$$R := \bigcup_{k=1}^{4}\left(\bigcap_{j\neq k}\{X_j \le t\}\right).$$

However, since the 4 events being "unioned" are not pairwise disjoint, it is more convenient to write R as the union of 5 pairwise disjoint events,

$$\begin{aligned} R = \bigcup_{i=1}^{4}\left[\{X_i > t\} \cap \left(\bigcap_{j\neq i}\{X_j \le t\}\right)\right] \\ \cup \left(\{X_1 \le t\} \cap \{X_2 \le t\} \cap \{X_3 \le t\} \cap \{X_4 \le t\}\right). \end{aligned}$$

(b) The corresponding probability is the sum of the probabilities of the 5 disjoint events:

$$\begin{aligned} \mathsf{P}(R) &= 4e^{-\lambda t}(1-e^{-\lambda t})^3 + (1-e^{-\lambda t})^4 \\ &= (1-e^{-\lambda t})^3[4e^{-\lambda t} + (1-e^{-\lambda t})] \\ &= (1-e^{-\lambda t})^3(1+3e^{-\lambda t}). \end{aligned}$$

[1] This was not the case for the Morandi bridge, which collapsed when the first cable stay failed. See https://www.pbs.org/wgbh/nova/video/why-bridges-collapse/?linkId=74764116

3. [20 pts.] Suppose $Z = X + Y$, where X and Y are independent $\exp(\lambda)$ random variables. Find a formula for the conditional density $f_{Y|Z}(y|z)$ ***and*** specify the range of values of y and z where your density positive.

Solution. Begin by writing

$$f_{Y|Z}(y|z) = \frac{f_{YZ}(y,z)}{f_Z(z)} = \frac{f_{Z|Y}(z|y)f_Y(y)}{f_Z(z)}.$$

Since Z is the sum of i.i.d. $\exp(\lambda)$ random variables, we know from Problem 4.55(c) that Z is Erlang$(2,\lambda)$, which means that $f_Z(z) = \lambda(\lambda z)e^{-\lambda z}$ for $z \geq 0$. Next, we find $f_{Z|Y}(z|y)$ using substitution and independence. Write

$$\begin{aligned} F_{Z|Y}(z|y) = \mathsf{P}(Z \leq z|Y = y) = \mathsf{P}(X+Y \leq z|Y = y) &= \mathsf{P}(X + y \leq z|Y = y) \\ &= \mathsf{P}(X \leq z - y|Y = y) \\ &= \mathsf{P}(X \leq z - y) = F_X(z-y). \end{aligned}$$

It follows that $f_{Z|Y}(z|y) = f_X(z-y)$. Putting this all together, we have

$$f_{Y|Z}(y|z) = \frac{f_X(z-y)f_Y(y)}{f_Z(z)} = \frac{\lambda e^{-\lambda(z-y)} \cdot \lambda e^{-\lambda y}}{\lambda(\lambda z)e^{-\lambda z}} = \frac{1}{z}, \quad 0 \leq y \leq z, \text{and } z > 0,$$

where the bounds on y come from the facts that $f_X(z-y) = 0$ for $z - y \leq 0$, $f_Y(y) = 0$ for $y \leq 0$, and $f_Z(z) = 0$ for $z \leq 0$. Notice that $f_{Y|Z}(\cdot|z) \sim \text{uniform}[0,z]$, and for future reference, $\mathsf{E}[Y|Z = z] = z/2$. How does this compare with the linear MMSE estimator of Y based on Z?

PART 2 — More Abstract

4. [20 pts.] Let $u_1, \ldots, u_n$ be an orthonormal basis for $\mathbb{R}^n$, and let Z be a zero mean random variable with variance σ^2. Put $X := Zu_1$. Find the "ingredients" of the Karhunen–Loève expansion of the random vector X; i.e., find a diagonal matrix Λ and a matrix P such $P' \operatorname{cov}(X) P = \Lambda$ and $P'P = I$. **Justify your answer.**

Solution. First observe that $\operatorname{cov}(X) = \mathsf{E}[XX'] = \mathsf{E}[Zu_1u_1'Z' = \mathsf{E}[Z^2]u_1u_1' = \sigma^2 u_1u_1'$. In the diagonalization of a symmetric matrix $C = P\Lambda P'$, if we let p_k denote the kth column of P, then we can also write

$$C = \sum_{k=1}^{n} \lambda_k p_k p_k'.$$

With this observation, the problem can be solved with $\Lambda = \operatorname{diag}(\sigma^2, 0, \ldots, 0)$ and $P = [u_1|\cdots|u_n]$.

Alternative Solution. We begin with $\operatorname{cov}(X) = \sigma^2 u_1u_1'$. Since we know that the columns of P must be orthonormal, this suggests that we take P to be the matrix with columns $u_1, \ldots, u_n$, and observe that $u_1'P = [1, 0, \ldots, 0]$. Then

$$P' \operatorname{cov}(X) P = \sigma^2 P' u_1 u_1' P = \sigma^2 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix} =: \Lambda.$$

Since $\operatorname{cov}(X)$, P, and Λ satisfy the required relationships, it follows that $Y = P'X = P'Zu_1 = ZP'u_1 = [Z, 0, \ldots, 0]'$; i.e., $Y_1 = Z$ and $Y_i = 0$ for $i = 2, \ldots, n$. We also have $PY = Zu_1 = X$. In other words, X was given in the form of the Karhunen–Loève expansion.

5. [20 pts.] Consider the problem of estimating a random variable X based on repeated noisy measurements $Y_i = X + V_i$. Suppose that $X, V_1, \ldots, V_n$ are uncorrelated random variables, with the V_i all having mean zero and variance σ_V^2, and with X having mean m_X and variance σ_X^2. Since X is a scalar, the linear MMSE estimate of X based on the vector $Y := [Y_1, \ldots, Y_n]'$ has the form $A(Y - m_Y) + m_X$ where A is a **row vector**, say $A = [A_1, \ldots, A_n]$. Find A.

Solution. If we let $\mathbf{1}$ denote the column vector of all ones, then $Y = X\mathbf{1} + V$, $\mathsf{E}[Y] = m_X\mathbf{1}$, $Y - m_Y = (X - m_X)\mathbf{1} + V$, $C_{XY} = \mathsf{E}[(X - m_X)'\{(X - m_X)\mathbf{1} + V\}'] = \sigma_X^2\mathbf{1}'$, and

$$C_Y = \mathsf{E}[\{(X - m_X)\mathbf{1} + V\}\{(X - m_X)\mathbf{1} + V\}'] = \sigma_X^2\mathbf{1}\mathbf{1}' + \sigma_V^2 I.$$

Since C_Y is positive definite, $A = C_Y^{-1}C_{XY}$. However, an analysis of the equation $AC_Y = C_{XY}$ in component form (see below) shows that all the components of the row vector A have to be the same, and their common value is $\sigma_X^2/[n\sigma_X^2 + \sigma_V^2]$.

Alternative Solution. Fortunately, the component form of $AC_Y = C_{XY}$ is easy to obtain without introducing $\mathbf{1}$. First note that since $\mathsf{E}[Y_i] = m_X$, we have $Y_i - \mathsf{E}[Y_i] = (X + V_i) - m_X = (X - m_X) + V_i$. Hence,

$$(C_Y)_{ij} = \mathsf{E}[\{(X - m_X) + V_i\}\{(X - m_X) + V_j\}] = \sigma_X^2 + \sigma_V^2\delta_{ij}.$$

Next,

$$(C_{XY})_j = \mathsf{E}[(X - m_X)\{(X - m_X) + V_j\}] = \sigma_X^2.$$

Then the component form of $AC_Y = C_{XY}$ is $(AC_Y)_j = (C_{XY})_j$. The left-hand side is

$$\sum_{i=1}^n A_i(C_Y)_{ij} = \sum_{i=1}^n A_i(\sigma_X^2 + \sigma_V^2\delta_{ij}) = \left(\sum_{i=1}^n A_i\right)\sigma_X^2 + A_j\sigma_V^2.$$

Now suppose we have solved

$$\left(\sum_{i=1}^n A_i\right)\sigma_X^2 + A_j\sigma_V^2 = \sigma_X^2, \quad j = 1, \ldots, n.$$

Putting $s := \sum_{i=1}^n A_i$ the above equations become $s\sigma_X^2 + A_j\sigma_V^2 = \sigma_X^2$, or $A_j = \sigma_X^2/[s\sigma_X^2 + \sigma_V^2]$. In other words, if there is a solution, the A_j must all be the same, say $A_j = a$. Then the above display becomes $na\sigma_X^2 + a\sigma_V^2 = \sigma_X^2$, and it follows that $a = \sigma_X^2/[n\sigma_X^2 + \sigma_V^2]$.

We can now write the linear MMSE estimate as

$$\begin{aligned} A(Y - m_Y) + m_X = AY + (m_X - Am_Y) &= \begin{bmatrix} a \cdots a \end{bmatrix} Y + m_X - am_Y \\ &= a\sum_{i=1}^n Y_i + (1 - na)m_X \\ &= \frac{\sigma_X^2}{n\sigma_X^2 + \sigma_V^2}\sum_{i=1}^n Y_i + \frac{\sigma_V^2}{n\sigma_X^2 + \sigma_V^2}. \end{aligned}$$

For large n, the estimate is approximately equal to the sample mean. This can be seen by writing

$$\begin{aligned} A(Y - m_Y) + m_X &= \underbrace{\frac{n\sigma_X^2}{n\sigma_X^2 + \sigma_V^2}}_{\approx 1 \text{ for large } n} \cdot \frac{1}{n}\sum_{i=1}^{n} Y_i + \underbrace{\frac{\sigma_V^2}{n\sigma_X^2 + \sigma_V^2}}_{\approx 0 \text{ for large } n} \\ &\approx \frac{1}{n}\sum_{i=1}^{n} Y_i. \end{aligned}$$