ECE 730, Lec. 1
Final Exam
Monday, 16 Dec. 2019
12:25 pm – 2:25 pm
2540 EH

100 Points

Justify your answers! **Be precise!**

Closed Book **Closed Notes** **No Calculators**

You may bring two sheets of 8.5$\times$11 paper with notes written on both sides.

1. [15 pts] Suppose $X \sim \exp(\lambda)$ and $Y \sim \exp(\mu)$, where X and Y are independent. Compute $\mathsf{E}[(X+Y)^2]$.

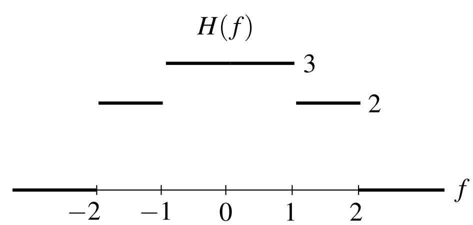
   Solution. Begin with

   $$\mathsf{E}[(X+Y)^2] = \mathsf{E}[X^2+2XY+Y^2] = \mathsf{E}[X^2]+2\mathsf{E}[X]\mathsf{E}[Y]+\mathsf{E}[Y^2],$$

   where we have used the linearity of expectation and the independence of X and Y. Using the tables, $\mathsf{E}[X^2] = 2/\lambda^2$, $\mathsf{E}[X] = 1/\lambda$, $\mathsf{E}[Y] = 1/\mu$, and $\mathsf{E}[Y^2] = 2/\mu^2$. Putting this all together, we have

   $$\mathsf{E}[(X+Y)^2] = 2/\lambda^2 + 2(1/\lambda)(1/\mu) + 2/\mu^2 = 2\big[1/\lambda^2 + 1/(\lambda\mu) + 1/\mu^2\big].$$

2. [15 pts] White noise with power spectral density $S_X(f) = N_0/2$ is applied to the lowpass filter $H(f)$ shown below.



   If the system output is denoted by Y_t, find the expected instantaneous output power $\mathsf{E}[Y_t^2]$.

   Solution. Write

   $$\mathsf{E}[Y_t^2] = \int_{-\infty}^{\infty} S_Y(f)\,df = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\,df = \int_{-\infty}^{\infty} |H(f)|^2 (N_0/2)\,df = (9\cdot 2+4\cdot 2)N_0/2 = 13N_0.$$

3. [15 pts] Let X_n converge in probability to X, where $X \sim \text{Laplace}(\lambda)$.

   (a) Determine whether or not

   $$\cos(X_n) \text{ converges in probability to } \cos(X).$$

   Justify your answer.

   (b) Determine whether or not

   $$\lim_{n\to\infty} \mathsf{E}[\cos(X_n)] = \mathsf{E}[\cos(X)].$$

   Justify your answer.

   (c) Evaluate $\mathsf{E}[\cos(X)]$. *Hint:* Don't compute any integrals.

   Solution.

   (a) Yes, continuous functions preserve convergence in probability.

   (b) Since convergence in probability implies convergence in distribution, and since cos is bounded and continuous, the equation holds.

(c) Write

$$\mathsf{E}[\cos(X)] = \operatorname{Re}\mathsf{E}[e^{j\nu X}]\big|_{\nu=1} = \operatorname{Re}\frac{\lambda^2}{\lambda^2-(j\nu)^2}\bigg|_{\nu=1} = \frac{\lambda^2}{\lambda^2+1}.$$

4. [15 pts.] Give an example of a Gaussian random vector $[X,Y]'$ that does *not* have a joint density, but for which at least one component does have a marginal density.

Solution. Let $X \sim N(0,1)$, and put $Y := 2X$. Then the covariance matrix is

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

which has zero determinant. Note also that $c_1X + c_2Y = (c_1+2c_2)X \sim N(0,(c_1+2c_2)^2)$, which show shows that $[X,Y]'$ is a Gaussian random vector.

Alternative Solution. Let $X \sim N(0,1)$ and let Y be a constant random variable with value y. Then the covariance matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which has zero determinant. Also, $c_1X + c_2y \sim N(c_2y, c_1^2)$.

5. [20 pts.] Let $U \sim \text{uniform}[-2,2]$, and put $X_n := (4-U^2)^n$. Let $G := \{X_n \to 0\}$.
   (a) Compute $\mathsf{P}(G)$.
   (b) Does X_n converge almost surely to 0? **Justify your answer.**

Solution.

(a) Since $4-U^2 \ge 0$, $X_n = (4-U^2)^n \to 0$ if and only if $4-U^2 < 1$, which happens if and only if $3 < U^2$, or $\sqrt{3} < |U|$.

$$\mathsf{P}(G) = \mathsf{P}(|U| > \sqrt{3}) = 2(2-\sqrt{3})/4 = (2-\sqrt{3})/2 = 1-\sqrt{3}/2.$$

(b) It is easy to check that $\mathsf{P}(G) < 1$, which implies X_n does *not* converge almost surely to 0.

Alternative Solution. If we can show that X_n does not converge in probability to zero, then we will know that X_n does not converge almost surely to zero. Fix any $\varepsilon > 0$ and write

$$\begin{aligned}
\mathsf{P}(|X_n| \ge \varepsilon) &= \mathsf{P}(X_n \ge \varepsilon), \qquad \text{since } X_n \ge 0,\\
&= \mathsf{P}((4-U^2)^n \ge \varepsilon)\\
&= \mathsf{P}(4-U^2 \ge \varepsilon^{1/n})\\
&= \mathsf{P}(4-\varepsilon^{1/n} \ge U^2)\\
&= 1-\mathsf{P}(|U| \le \sqrt{4-\varepsilon^{1/n}}\,)\\
&= 1-\tfrac{1}{2}\sqrt{4-\varepsilon^{1/n}} \to 1-\sqrt{3}/2 > 0,
\end{aligned}$$

where the last two lines assume n is large; i.e., since $\varepsilon^{1/n} = \exp(\frac{1}{n}\log\varepsilon) \to \exp(0) = 1$, for large n, $\varepsilon^{1/n} < 4$, and $\sqrt{4-\varepsilon^{1/n}} < 2$. This is important because $U \sim \text{uniform}[-2,2]$.

6. [20 pts.] Suppose X_n converges in probability to X. Suppose also that B is a positive, finite constant such that $|X_n| \leq B$ and $|X| \leq B$. Determine whether or not X_n converges in mean of order 2 to X. **Justify your answer.**

Solution. Let $\varepsilon > 0$ be given, and write

$$\begin{aligned}\mathsf{E}[|X_n - X|^2] &= \mathsf{E}[|X_n - X|^2 \mathbf{1}_{\{|X_n - X| \geq \varepsilon\}}] + \mathsf{E}[|X_n - X|^2 \mathbf{1}_{\{|X_n - X| < \varepsilon\}}] \\ &\leq 4B^2\, \mathsf{P}(|X_n - X| \geq \varepsilon) + \varepsilon^2\, \mathsf{P}(|X_n - X| < \varepsilon) \\ &\leq 4B^2\, \mathsf{P}(|X_n - X| \geq \varepsilon) + \varepsilon^2.\end{aligned}$$

For sufficiently large n, the last probability is less than $\varepsilon/(4B^2)$, and so

$$\mathsf{E}[|X_n - X|^2] \leq \varepsilon + \varepsilon^2.$$

Since ε was arbitrary, $\mathsf{E}[|X_n - X|^2] \to 0$.

Alternative Solution. Since $|X_n - X| \leq |X_n| + |X| \leq 2B$, we introduce the bounded continuous function

$$g(t) := \begin{cases} t, & 0 \leq t \leq 2B, \\ 0, & \text{otherwise.} \end{cases}$$

Then $|X_n - X| = g(|X_n - X|)$. Of course, $g(t)^2$ is also bounded and continuous. Since $|X_n - X| \to 0$ almost surely, it also converges to zero in distribution. Hence,

$$\mathsf{E}[|X_n - X|^2] = \mathsf{E}[g(|X_n - X|)^2] \to \mathsf{E}[g(0)^2] = 0.$$