

Discrete random variables

Bernoulli(p)

$$P(X = 1) = p, \quad P(X = 0) = 1 - p.$$

$$E[X] = p, \quad \text{var}(X) = p(1 - p), \quad G_X(z) = (1 - p) + pz.$$

Poisson(λ)

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, \dots$$

$$E[X] = \lambda, \quad \text{var}(X) = \lambda, \quad G_X(z) = e^{\lambda(z-1)}.$$

geometric₀(p)

$$P(X = k) = (1 - p)p^k, \quad k = 0, 1, 2, \dots$$

$$E[X] = \frac{p}{1 - p}, \quad \text{var}(X) = \frac{p}{(1 - p)^2}, \quad G_X(z) = \frac{1 - p}{1 - pz}.$$

geometric₁(p)

$$P(X = k) = (1 - p)p^{k-1}, \quad k = 1, 2, 3, \dots$$

$$E[X] = \frac{1}{1 - p}, \quad \text{var}(X) = \frac{p}{(1 - p)^2}, \quad G_X(z) = \frac{(1 - p)z}{1 - pz}.$$

binomial(n, p)

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, \dots, n.$$

$$E[X] = np, \quad \text{var}(X) = np(1 - p), \quad G_X(z) = [(1 - p) + pz]^n.$$

negative binomial or Pascal(m, p)

$$P(X = k) = \binom{k-1}{m-1} (1 - p)^m p^{k-m}, \quad k = m, m+1, \dots$$

$$E[X] = \frac{m}{1 - p}, \quad \text{var}(X) = \frac{mp}{(1 - p)^2}, \quad G_X(z) = \left[\frac{(1 - p)z}{1 - pz} \right]^m.$$

Note that Pascal(1, p) is the same as geometric₁(p).

Note. To obtain the moment generating function from $G_X(z)$, let $z = e^s$. To obtain the characteristic function, let $z = e^{jv}$. In other words, $M_X(s) = G_X(e^s)$ and $\varphi_X(v) = G_X(e^{jv})$.

Fourier transforms

Fourier transform

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt$$

Inversion formula

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{j2\pi ft} df$$

$h(t)$	$H(f)$
$I_{[-T, T]}(t)$	$2T \frac{\sin(2\pi T f)}{2\pi T f}$
$2W \frac{\sin(2\pi W t)}{2\pi W t}$	$I_{[-W, W]}(f)$
$(1 - t /T) I_{[-T, T]}(t)$	$T \left[\frac{\sin(\pi T f)}{\pi T f} \right]^2$
$W \left[\frac{\sin(\pi W t)}{\pi W t} \right]^2$	$(1 - f /W) I_{[-W, W]}(f)$
$e^{-\lambda t} u(t)$	$\frac{1}{\lambda + j2\pi f}$
$e^{-\lambda t }$	$\frac{2\lambda}{\lambda^2 + (2\pi f)^2}$
$\frac{\lambda}{\lambda^2 + t^2}$	$\pi e^{-2\pi \lambda f }$
$e^{-(t/\sigma)^2/2}$	$\sqrt{2\pi} \sigma e^{-\sigma^2 (2\pi f)^2/2}$

Note. The indicator function $I_{[a, b]}(t) := 1$ for $a \leq t \leq b$ and $I_{[a, b]}(t) := 0$ otherwise. In particular, $u(t) := I_{[0, \infty)}(t)$ is the unit step function.

Series formulas

$$\sum_{k=0}^{N-1} z^k = \frac{1 - z^N}{1 - z}, \quad z \neq 1 \quad \left| \quad e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \right| \quad (a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}, \quad |z| < 1 \quad \left| \quad \lim_{n \rightarrow \infty} \left(1 + \frac{z}{n} \right)^n = e^z \quad \right| \quad \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

Continuous random variables

uniform[a, b]

$$f_X(x) = \frac{1}{b-a} \quad \text{and} \quad F_X(x) = \frac{x-a}{b-a}, \quad a \leq x \leq b.$$

$$E[X] = \frac{a+b}{2}, \quad \text{var}(X) = \frac{(b-a)^2}{12}, \quad M_X(s) = \frac{e^{sb} - e^{sa}}{s(b-a)}.$$

exponential, $\exp(\lambda)$

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{and} \quad F_X(x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

$$E[X] = 1/\lambda, \quad \text{var}(X) = 1/\lambda^2, \quad E[X^n] = n!/\lambda^n.$$

$$M_X(s) = \lambda/(\lambda - s), \quad \text{Re } s < \lambda.$$

Laplace(λ)

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}.$$

$$E[X] = 0, \quad \text{var}(X) = 2/\lambda^2. \quad M_X(s) = \lambda^2/(\lambda^2 - s^2), \quad -\lambda < \text{Re } s < \lambda.$$

Cauchy(λ)

$$f_X(x) = \frac{\lambda/\pi}{\lambda^2 + x^2}, \quad F_X(x) = \frac{1}{\pi} \tan^{-1}\left(\frac{x}{\lambda}\right) + \frac{1}{2}.$$

$$E[X] = \text{undefined}, \quad E[X^2] = \infty, \quad \varphi_X(v) = e^{-\lambda|v|}.$$

Odd moments are not defined; even moments are infinite. Since the first moment is not defined, central moments, including the variance, are not defined.

Gaussian or normal, $N(m, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]. \quad F_X(x) = \text{normcdf}(x, m, \text{sigma}).$$

$$E[X] = m, \quad \text{var}(X) = \sigma^2, \quad E[(X-m)^{2n}] = 1 \cdot 3 \cdots (2n-3)(2n-1)\sigma^{2n} = \frac{(2n)!}{2^n n!} \sigma^{2n},$$

$$M_X(s) = e^{sm + s^2 \sigma^2 / 2}.$$

Rayleigh(λ)

$$f_X(x) = \frac{x}{\lambda^2} e^{-(x/\lambda)^2/2} \quad \text{and} \quad F_X(x) = 1 - e^{-(x/\lambda)^2/2}, \quad x \geq 0.$$

$$E[X] = \lambda\sqrt{\pi/2}, \quad E[X^2] = 2\lambda^2, \quad \text{var}(X) = \lambda^2(2 - \pi/2).$$

$$E[X^n] = 2^{n/2} \lambda^n \Gamma(1 + n/2).$$

Note. To obtain the characteristic function from $M_X(s)$, let $s = jv$. In other words, $\varphi_X(v) = M_X(jv)$.

Continuous random variables

gamma(p, λ)

$$f_X(x) = \lambda \frac{(\lambda x)^{p-1} e^{-\lambda x}}{\Gamma(p)}, \quad x > 0, \quad \text{where } \Gamma(p) := \int_0^\infty x^{p-1} e^{-x} dx, \quad p > 0.$$

Recall that $\Gamma(p) = (p-1) \cdot \Gamma(p-1)$, $p > 1$.

$$F_X(x) = \text{gamcdf}(x, p, 1/\lambda).$$

$$E[X^n] = \frac{\Gamma(n+p)}{\lambda^n \Gamma(p)}. \quad M_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^p, \quad \text{Re } s < \lambda.$$

Note that gamma(1, λ) is the same as exp(λ).

Erlang(m, λ) := gamma(m, λ), $m = \text{integer}$

Since $\Gamma(m) = (m-1)!$

$$f_X(x) = \lambda \frac{(\lambda x)^{m-1} e^{-\lambda x}}{(m-1)!} \quad \text{and} \quad F_X(x) = 1 - \sum_{k=0}^{m-1} \frac{(\lambda x)^k}{k!} e^{-\lambda x}, \quad x > 0.$$

Note that Erlang(1, λ) is the same as exp(λ).

chi-squared(k) := gamma($k/2, 1/2$)

If k is an even integer, then chi-squared(k) is the same as Erlang($k/2, 1/2$).

Since $\Gamma(1/2) = \sqrt{\pi}$,

$$\text{for } k = 1, f_X(x) = \frac{e^{-x/2}}{\sqrt{2\pi x}}, \quad x > 0.$$

$$\text{Since } \Gamma\left(\frac{2m+1}{2}\right) = \frac{(2m-1) \cdots 5 \cdot 3 \cdot 1}{2^m} \sqrt{\pi},$$

$$\text{for } k = 2m+1, f_X(x) = \frac{x^{m-1/2} e^{-x/2}}{(2m-1) \cdots 5 \cdot 3 \cdot 1 \sqrt{2\pi}}, \quad x > 0.$$

$$F_X(x) = \text{chi2cdf}(x, k).$$

Note that chi-squared(2) is the same as exp(1/2).

Weibull(p, λ)

$$f_X(x) = \lambda p x^{p-1} e^{-\lambda x^p} \quad \text{and} \quad F_X(x) = 1 - e^{-\lambda x^p}, \quad x > 0.$$

$$E[X^n] = \frac{\Gamma(1+n/p)}{\lambda^{n/p}}.$$

Note that Weibull(2, λ) is the same as Rayleigh($1/\sqrt{2\lambda}$) and that Weibull(1, λ) is the same as exp(λ).