1. Let X denote the real vector space of all continuous, real-valued functions on the interval $[0, 1]$ with inner product

$$\langle x_1, x_2 \rangle := \int_0^1 x_1(t)x_2(t)dt, \qquad x_1, x_2 \in X.$$

Define the linear operator $A : X \to \mathbb{R}$ by

$$Ax := \int_0^1 t^2 x(t)dt.$$

Treat $\mathbb{R}$ as an inner product space with inner product $(y_1, y_2) := y_1 y_2$ (i.e., ordinary multiplication of real numbers).

   (a) [10 pts.] The adjoint operator $A^* : \mathbb{R} \to X$. Find a formula for $(A^* y)(t)$.

   (b) [5 pts.] When $y = 3$, sketch a graph of $(A^* y)(t)$.

2. [15 pts.] Let $X = L^2[0, 1]$ denote the set of real-valued, square-integrable functions on $[0, 1]$, equipped with the usual inner product, $\langle u, v \rangle := \int_0^1 u(t)v(t)\,dt$ for $u, v \in X$. Define $A: X \to X$ by

$$(Ax)(t) := \int_0^t x(\theta)\,d\theta, \quad 0 \le t \le 1.$$

Find $(A^* y)(\theta)$ for $\theta \in [0, 1]$.

3. [10 pts.] Let $X = Y =$ the set of all continuously differentiable, real-valued functions on $[0, T]$ that vanish at $t = 0$ and at $t = T$. Let the inner product be given by $\langle x_1, x_2 \rangle := \int_0^T x_1(t)x_2(t)\,dt$. Let $(Ax)(t) := \dot{x}(t)$, where $\dot{x}(t)$ is the usual derivative of $x(t)$ with respect to t. For $y \in Y$, find $(A^* y)(t)$.

4. Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Let M be a closed subspace of X. For $x \in X$, let $\hat{x}$ denote the projection of x onto M; i.e., $\hat{x}$ is the *unique* element of M such that $\|x - \hat{x}\| \le \|x - m\|$ for all $m \in M$.

   (a) [15 pts.] Define the mapping $P : X \to M$ by $P(x) := \hat{x}$. Show that P is a linear operator.

   (b) [15 pts.] Show that for all x and y in X, $\|\hat{x} - \hat{y}\| \le \|x - y\|$.

   (c) [20 pts.] If $y = x + Px$, solve for x in terms of y and Py.

5. [15 pts.] Let a and b be positive constants. Show that the ellipse

$$E := \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}$$

is a convex subset of $\mathbb{R}^2$.

6. [20 pts.] Let U be a $p \times m$ matrix, and let V be an $n \times q$ matrix. Assume that U' and V are nonsingular. Define the linear operator $A\colon \mathrm{I\!R}^{m\times n} \to \mathrm{I\!R}^{p\times q}$ by $AX := UXV$. Assume standard inner products.[1] Assuming that there are multiple solutions of $AX = Y_0$, find the one of minimum norm. Your answer should be in terms of U, V, and Y_0.

7. [20 pts.] You wish to approximate the function $x(t) = t^3$ for $0 \le t \le 1$ using a polynomial of degree 1; i.e., $\hat{x}(t) = c_1 + c_2 t$. Find numerical values of c_1 and c_2 that minimize the mean squared error,
$$\int_0^1 |x(t) - \hat{x}(t)|^2\,dt.$$

8. Let $X = C[0,2]$ denote the set of real-valued, continuous functions on $[0,2]$ equipped with the usual inner product, $\langle x, y\rangle := \int_0^2 x(t)y(t)\,dt$ for $x, y \in X$. Put $M := \{x \in X : x(t) = 0 \text{ for } 1 \le t \le 2\}$.
   (a) [15 pts.] Show that M is closed.
   (b) [15 pts.] Prove that $M^\perp = \{y \in X : y(t) = 0, \text{ for } 0 \le t \le 1\}$.
   (c) [10 pts.] Does $X = M \oplus M^\perp$? **Justify your answer!!!**

9. [20 pts.] Let C be a convex subset of a real vector space X. For any $n \ge 2$, show that if $x_1, \ldots, x_n$ are points in C, then
$$\sum_{i=1}^n \lambda_i x_i \in C,$$
whenever $\lambda_i \ge 0$ and $\sum_{i=1}^n \lambda_i = 1$.

10. Let $(X, \langle\cdot,\cdot\rangle)$ be a complex inner product space. Suppose that a linear operator $A\colon X \to X$ satisfies $\langle Ax, x\rangle \in \mathrm{I\!R}$ for all $x \in X$. Show that A is self adjoint.

[1] For $m \times n$ matrices X_1 and X_2, use $\langle X_1, X_2\rangle = \mathrm{tr}(X_1 X_2')$. For $p \times q$ matrices Y_1 and Y_2, use $(Y_1, Y_2) = \mathrm{tr}(Y_1 Y_2')$. Note that for any matrices S and T, where S is $\mu \times \nu$ and T is $\nu \times \mu$, we have $\mathrm{tr}(ST) = \mathrm{tr}(TS)$.