ECE 735

Fall 06, Exam 1 Review sol.

2-8 Let $Y_i(f) = \int_{-\infty}^{\infty} y_i(t) e^{-j2\pi f t} dt$

$$= H(f) X_i(f)$$
$$= \frac{1}{1+j2\pi f} X_i(f)$$

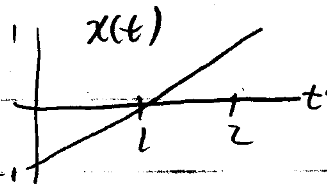
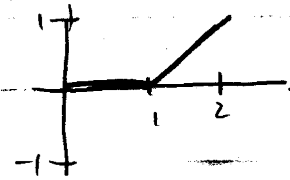
Suppose $\sum_{i=1}^{n} c_i y_i(t) = 0$. Then $\sum_{i=1}^{n} c_i Y_i(f) = 0$,

or

$$H(f) \sum_{i=1}^{n} c_i X_i(f) = 0$$

$H(f) \neq 0 \Rightarrow \sum_{i=1}^{n} c_i X_i(t) = 0 \Rightarrow \sum_{i=1}^{n} c_i x_i(t) = 0.$

Since the x_i are LI, the $c_i = 0$. $\therefore$ The $y_i(t)$ are LI.

6-3)  Claim $\hat{x}(t)$: , which

is continuous. Check that $\langle x-\hat{x}, y-\hat{x}\rangle \le 0 \;\forall y \in C$. Write

$$\langle x-\hat{x}, y-\hat{x}\rangle = \int_0^2 \underbrace{[x(t)-\hat{x}(t)]}_{=0 \text{ for } 1\le t\le 2}[y(t)-\hat{x}(t)]\,dt = \int_0^1 \underbrace{(t-1)}_{\substack{\le 0 \\ \text{for} \\ 0\le t\le 1}} \underbrace{y(t)}_{\substack{\ge 0 \text{ since} \\ y\in C}}\,dt \le 0.$$

6-34 observe that $\mathrm{Re}\langle x - P_K(P_M(x)), y - P_K(P_M(x))\rangle$, $y \in C = M \cap K \subset \begin{cases} M \\ K \end{cases}$

$$= \mathrm{Re}\langle \underbrace{x - P_M(x)}_{} + P_M(x) - P_K(P_M(x)), \underbrace{y - P_K(P_M(x))}_{\in M}\rangle$$

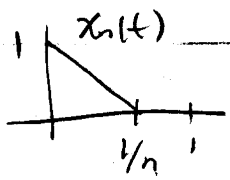
$$= \mathrm{Re}\langle P_M(x) - \underbrace{P_K(P_M(x))}_{\in K}, \underbrace{y}_{\in K} - P_K(P_M(x))\rangle \quad \underbrace{}_{\in M}$$

$\le 0.$

6-35 Let $x, y \in \bar{C}$. For $0 \le \lambda \le 1$, we must show that $\lambda x + (1-\lambda) y \in \bar{C}$. Since $x, y \in \bar{C}$, $\exists\, x_n, y_n \in C$ with $x_n \to x$, $y_n \to y$. Then

$$\lambda x + (1-\lambda) y = \lambda \lim_n x_n + (1-\lambda) \lim y_n$$
$$= \lim \underbrace{(\lambda x_n + (1-\lambda) y_n)}_{\in C} \in \bar{C}.$$

7-3 Let $x(t) := 0$ for $0 \le t \le 1$. Then x is continuous, and $x \in \mathbb{X}$. Clearly



$$\|x_n - x\| = \int_0^1 |x_n(t) - x(t)|\, dt$$
$$= \int_0^1 x_n(t)\, dt = \frac{1}{2n} \to 0,$$

but $f(x_n) = x_n(0) = 1 \not\to f(x) = x(0) = 0$,

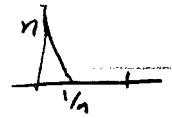
7-4 (a) For fixed n,

$$|f_n(x)| = \left| n \int_0^{1/n} x(\tau)\, d\tau \right| \le n \int_0^1 |x(\tau)|\, d\tau = n\|x\|$$

+ so f_n is a bounded linear functional.

(b) If we put $X(t) := \int_0^t x(\tau)\, d\tau$, then

$$f_n(x) = \frac{X(1/n) - X(0)}{1/n} \to X'(0) = x(0) = f(x)$$

If f were bounded, we could write $|f(x)| \le B\|x\|$ for some finite B. But consider $x_n(t) =$  Then $\|x_n\| = 1/2$, but $|f(x_n)| = x_n(0) = n$

$\not\le B\|x_n\| = B/2$ for large n.

$\therefore f$ is not bounded.

7-6 Let $\underline{w}_1,\dots,\underline{w}_d$ be a basis for W. If $\underline{w}=\sum_{i=1}^{d} c_i \underline{w}_i$, then

$$|f(\underline{w})| = \left|\sum_{i=1}^{d} c_i f(\underline{w}_i)\right| \le \sum_{i=1}^{n} |c_i|\,|f(\underline{w}_i)|$$

$$\le \|c\|_\infty \sum_{i=1}^{d} |f(\underline{w}_i)|$$

$$\le \frac{\|\underline{w}\|}{K_1} \sum_{i=1}^{d} |f(\underline{w}_i)| = B\|\underline{w}\|$$

where $B := \frac{1}{K_1}\sum_{i=1}^{d} |f(\underline{w}_i)|$.

7-39 $A^*A = I \Rightarrow A$ is nonsingular. By Prop. 7.12, A is onto. Then by Lemma 7.22, $AA^* = I$.

7-43. $A = [1 \;\; 2]$ $\quad A^* = \begin{bmatrix}1\\2\end{bmatrix}$ $\quad AA^* = [1 \;\; 2]\begin{bmatrix}1\\2\end{bmatrix} = 5$

$$\tilde{x}_0 = A^*(AA^*)^{-1}y = \begin{bmatrix}1/5\\2/5\end{bmatrix} y$$

7-44 We must first find A^*. Write

$$(AX, Z) = (UXV, Z) = tr(UXVZ') = tr(XVZ'U)$$
$$= tr(X[U'ZV']')$$
$$= \langle X, A^*Z\rangle$$

$\therefore A^*Z = U'ZV'$. We must now solve $A(A^*Z) = Y_0$ or $U[U'ZV']V = Y_0$. $\therefore$ $Z = (UU')^{-1}Y_0(V'V)^{-1}$. Thus $A^*Z\ U'(UU')^{-1}Y_0(V'V)^{-1}V'$ is the desired minimum norm solution.

7-45 We must find the adjoint again. Write

$$(AX, Z) = (UXV, Z) = tr(UXVZ')$$
$$= tr(XVZ'U)$$
$$= tr(XQQ^{-1}VZ'U)$$
$$= tr(XQ[U'ZV'Q^{-1}]')$$
$$= \langle X, A^*Z \rangle$$

$\therefore\ A^*Z = U'ZV'Q^{-1}$

Now solve $AA^*Z = Y_0$ or $UU'ZV'Q^{-1}V = Y_0$

So $Z = (UU')^{-1}Y_0(V'Q^{-1}V)^{-1}$

& the minimum-norm solution is

$$A^*Z = U'(UU')^{-1}Y_0(V'Q^{-1}V)^{-1}V'Q^{-1}$$

Note that $V'Q^{-1}V$ is invertible since:

Let $B: \mathbb{R}^q \to \mathbb{R}^n$ be defined by $Bx = Vx$. Use the standard inner product on $\mathbb{R}^q$ & on $\mathbb{R}^n$ by $(y_1, y_2) = y_2'Q^{-1}y_1$

Then to find B^* write $(Bx, y) = y'Q^{-1}Vx = (V'Q^{-1}y)'x = \langle x, B^*y \rangle$,

i.e., $B^* = V'Q^{-1}$. So $B^*B = V'Q^{-1}V$. Since V is nonsingular, we have $\ker V'Q^{-1}V = \ker B^*B = \ker B = \ker V = \{0\}$.

7-46 This is really a minimum-norm solution problem phrased in different terminology. So, $\tilde{x}_0 = A^*(AA^*)^{-1}y_0$.

If we change the norm on $\mathcal{X}$, the new solution is

$\tilde{x}_0 = Q^{-1}A^*(AQ^{-1}A^*)^{-1}y_0$.