Theorem 8.1. *Let X and Y be normed vector spaces, and let $A_N : X \to Y$ be a sequence of compact linear operators such that A_N converges to some limiting operator A. If Y is complete, then A is compact.*

Proof. Let $\|x_n\| = 1$. We must prove that there exists a subsequence $m_n \to \infty$ such that Ax_{m_n} converges. The main idea of the proof is to use a diagonal argument. Since A_1 is compact, there is a subsequence $m_n^1 \to \infty$ and a vector $y^1 \in Y$ such that

$$A_1 x_{m_n^1} \to y^1.$$

Next, there is a further subsequence m_n^2 and a vector y^2 such that

$$A_2 x_{m_n^2} \to y^2.$$

Since $\{m_n^2\}_{n=1}^\infty \subset \{m_n^1\}_{n=1}^\infty$,

$$A_1 x_{m_n^2} \to y^1.$$

Since Y is complete, it is enough to show that $Ax_{m_n^n}$ is Cauchy. Let $\varepsilon > 0$ be given. Since $A_N \to A$, there is a positive integer N such that

$$\|A_N - A\| < \varepsilon/3.$$

Now, for $k \geq N$,

$$\{m_n^k\}_{n=1}^\infty \subset \{m_n^N\}_{n=1}^\infty.$$

In particular,

$$m_k^k \in \{m_n^N\}_{n=1}^\infty.$$

Hence, $A_N x_{m_k^k} \to y^N$, and is therefore Cauchy. It follows that for $k \geq N$ and $i \geq N$ and both large enough,

$$\|A_N x_{m_k^k} - A_N x_{m_i^i}\| < \varepsilon/3.$$

Now observe that $\|Ax_{m_k^k} - Ax_{m_i^i}\|$ is upper bounded by

$$\|Ax_{m_k^k} - A_N x_{m_k^k}\| + \|A_N x_{m_k^k} - A_N x_{m_i^i}\| + \|A_N x_{m_i^i} - Ax_{m_i^i}\|.$$

The middle term is upper bounded by $\varepsilon/3$. The first and last terms are upper bounded by

$$\|A - A_N\| \quad \text{and} \quad \|A_N - A\|,$$

respectively. Since each of these is also upper bounded by $\varepsilon/3$, we are done. □