Example 2.8 (Methods of Proving Linear Independence). Let $x(t) = 1/t$ and $y(t) = 1/t^2$ for $0 < t < 1$. We show that x and y are linearly independent. Suppose that

$$\frac{c_1}{t} + \frac{c_2}{t^2} = 0, \quad t \in (0,1). \tag{2.2}$$

Since (2.2) holds for *all* $t \in (0,1)$, it must hold for any particular values of t, say $t = 1/4$ and $t = 1/2$. This leads to the system of equations

$$\begin{aligned} 4c_1 + 16c_2 &= 0, \\ 2c_1 + 4c_2 &= 0, \end{aligned}$$

which can easily be solved to show that $c_2 = 0$ and then that $c_1 = 0$ as well. Consider, however, another approach. Multiply (2.2) by t^2 to get

$$c_1 t + c_2 = 0, \quad t \in (0,1). \tag{2.3}$$

Since both sides are continuous functions of t, we can let $t \to 0$ to learn that $c_2 = 0$. Hence, $c_1 t = 0$ for $t \in (0,1)$. Specializing to $t = 1/2$ shows that $c_1 = 0$ as well. For a third approach, which avoids taking an explicit limit in (2.3), let us differentiate (2.3) instead. This yields $c_1 = 0$. Using this in (2.3) yields $c_2 = 0$ as well. We now combine these last two approaches to apply the linear functional method of Lemma 1.9. For $\varphi \in \operatorname{span}\{x, y\}$, put

$$\begin{aligned} \beta_1(\varphi) &:= \frac{\partial}{\partial t}\big[t^2\varphi(t)\big], \\ \beta_2(\varphi) &:= \lim_{t\to 0}\big[t^2\varphi(t)\big]. \end{aligned}$$

Then $\beta_1(x) = 1$, $\beta_1(y) = 0$, $\beta_2(x) = 0$, and $\beta_2(y) = 1$.

Remark. The preceding example shows that there are many approaches to proving that a collection of waveforms is linearly independent. In specific applications, one approach may be significantly easier to carry out than another.