Theorem 4.19. *A set E in a metric space is closed $\Leftrightarrow$ every converging sequence of points in E converges to a point in E.*

Proof. ($\Rightarrow$): Let E be closed. We must show that if $x_n \to x$ with $x_n \in E$, then $x \in E$. For a proof by contradiction, suppose otherwise that there is some sequence $x_n \in E$ that converges to a limit $x \notin E$. Then $x \in E^c$ where E^c is open. Hence, there is an $\varepsilon > 0$ with $B(x, \varepsilon) \subset E^c$. However, since $x_n \to x$, for large n $x_n \in B(x, \varepsilon) \subset E^c$. For these n, $x_n \in E^c$ and $x_n \in E$, which is a contradiction.

($\Leftarrow$): Suppose that every converging sequence from E has its limit in E. We must prove that E is closed. For a proof by contradiction, suppose otherwise that E is not closed. Then E^c is not open. Hence, there is an $x \in E^c$ such that there is no open ball about x contained in E^c. This implies that for each open ball of the form $B(x, 1/n)$, $B(x, 1/n) \not\subset E^c$; i.e., there is an $x_n \in B(x, 1/n) \cap E$. In other words, $x_n \in E$ and $\rho(x_n, x) < 1/n \to 0$; i.e., x_n is a sequence in E that converges to a point $x \notin E$. This contradicts the original assumption that every converging sequence from E has its limit it E. $\square$

Theorem 4.20 (Approximation). *Given $x \in \overline{E}$, either $x \in E$, or if $x \notin E$, we can approximate x by some $y \in E$. More precisely, given $\varepsilon > 0$, there is a $y \in E$ with $\rho(x, y) < \varepsilon$. Hence, by taking $\varepsilon = 1/n$, there is an $x_n \in E$ with $\rho(x_n, x) < 1/n$. In other words, there is a sequence from E that converges to x.*

Proof. Let $x \in \overline{E}$ with $x \notin E$. We need to show that for every $\varepsilon > 0$, there is a $y \in E$ with $y \in B(x, \varepsilon)$. Suppose otherwise that this is not the case. Then for some $\varepsilon > 0$, $B(x, \varepsilon) \cap E = \varnothing$. Equivalently, $E \subset B(x, \varepsilon)^c$, which is a closed set. Hence,

$$x \in \overline{E} = \bigcap_{\substack{C: E \subset C \text{ and} \\ C \text{ is closed}}} C \subset B(x, \varepsilon)^c.$$

Of course, $x \in B(x, \varepsilon)^c$ is a contradiction. $\square$