Theorem 4.19. A set $E$ in a metric space is closed $\iff$ every converging sequence of points in $E$ converges to a point in $E$.

Proof. ($\Rightarrow$): Let $E$ be closed. We must show that if $x_n \to x$ with $x_n \in E$, then $x \in E$. For a proof by contradiction, suppose otherwise that there is some sequence $x_n \in E$ that converges to a limit $x \notin E$. Then $x \in E^c$ where $E^c$ is open. Hence, there is an $\epsilon > 0$ with $B(x, \epsilon) \subset E^c$. However, since $x_n \to x$, for large $n$, $x_n \in B(x, \epsilon) \subset E^c$. For these $n$, $x_n \in E^c$ and $x_n \in E$, which is a contradiction.

($\Leftarrow$): Suppose that every converging sequence from $E$ has its limit in $E$. We must prove that $E$ is closed. For a proof by contradiction, suppose otherwise that $E$ is not closed. Then $E^c$ is not open. Hence, there is an $x \in E^c$ such that there is no open ball about $x$ contained in $E^c$. This implies that for each open ball of the form $B(x, 1/n), B(x, 1/n) \notin E^c$; i.e., there is an $x_n \in B(x, 1/n) \cap E$. In other words, $x_n \in E$ and $\rho(x_n, x) < 1/n \to 0$; i.e., $x_n$ is a sequence in $E$ that converges to a point $x \notin E$. This contradicts the original assumption that every converging sequence from $E$ has its limit in $E$.

Theorem 4.20 (Approximation). Given $x \in E$, either $x \in E$, or if $x \notin E$, we can approximate $x$ by some $y \in E$. More precisely, given $\epsilon > 0$, there is a $y \in E$ with $\rho(x, y) < \epsilon$. Hence, by taking $\epsilon = 1/n$, there is an $x_n \in E$ with $\rho(x_n, x) < 1/n$. In other words, there is a sequence from $E$ that converges to $x$.

Proof. Let $x \in E$ with $x \notin E$. We need to show that for every $\epsilon > 0$, there is a $y \in E$ with $y \in B(x, \epsilon)$. Suppose otherwise that this is not the case. Then for some $\epsilon > 0$, $B(x, \epsilon) \cap E = \emptyset$. Equivalently, $E \subset B(x, \epsilon)^c$, which is a closed set. Hence,

$$x \in E = \bigcap_{C : E \subset C \text{ and } C \text{ is closed}} C \subset B(x, \epsilon)^c.$$ 

Of course, $x \in B(x, \epsilon)^c$ is a contradiction.