Problem 7–45. Let U be a $p \times m$ matrix, and let V be an $n \times q$ matrix. Assume that U' and V are nonsingular. Define the linear operator $A{:}\mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$ by $AX := UXV$. For $m \times n$ matrices X_1 and X_2, use $\langle X_1, X_2\rangle = \mathrm{tr}(X_1 X_2')$. For $p \times q$ matrices Y_1 and Y_2, use $(Y_1, Y_2) = \mathrm{tr}(Y_1 Y_2')$. Note that for any matrices S and T, where S is $\mu \times \nu$ and T is $\nu \times \mu$, we have $\mathrm{tr}(ST) = \mathrm{tr}(TS)$. Assuming that there are multiple solutions of $AX = Y_0$, find the one of minimum norm. Your answer should be in terms of U, V, and Y_0.

Problem 7–46. In the preceding problem suppose that $\langle X_1, X_2\rangle$ is redefined as $\langle X_1, X_2\rangle = \mathrm{tr}(X_1 Q X_2')$, where Q is a symmetric, positive-definite matrix. How does your answer change?

Problem 7–47. Given an operator $A{:}X \to Y$ and a vector y_0, solve the constrained minimization problem

$$\min \|x\|^2 \quad \text{subject to} \quad Ax = y_0.$$

Assume that $\|\cdot\|$ comes from an inner product $\langle\cdot,\cdot\rangle$ on X. Denote the inner product on Y by $(\cdot,\cdot)$. How does your answer change if $\|\cdot\|$ is replaced by $\|\cdot\|_Q$, where $\|\cdot\|_Q$ is the norm induced by the inner product $[\cdot,\cdot]$ defined in Problem 7–42?

We can recast Problem 7–46 as

$$\min\ \mathrm{tr}(XQX') \quad \text{subject to} \quad UXV = Y_0.$$

One instance of this problem arises as follows when p and q are both equal to m. Suppose we observe the noisy measurement $Vx + W$, where W is a n-dimensional white-noise vector with covariance matrix Q, V is a known gain matrix, and x is an unknown m-dimensional signal vector to be estimated. We estimate the signal by applying a matrix X to the measurement to obtain $X(Vx+W)$. The mean-squared error between the estimate and the signal is

$$\mathsf{E}[\|(XVx + XW) - x\|^2],$$

where in this expression $\|\cdot\|$ denotes the Euclidean norm on m-dimensional column vectors. We want to choose the $m \times n$ matrix X to minimize this expectation. However, the expectation depends on the unknown signal vector x. Writing the expectation as

$$\mathsf{E}[\|(XV - I)x + XW\|^2]$$

suggests that we restrict attention to matrices X such that $XV = I$. Then we just have to minimize

$$\begin{aligned}
\mathsf{E}[\|XW\|^2] &= \mathrm{tr}\big(\mathsf{E}[\|XW\|^2]\big) = \mathrm{tr}\big(\mathsf{E}[(XW)'(XW)]\big) \\
&= \mathsf{E}[\mathrm{tr}\{(XW)'(XW)\}] = \mathsf{E}[\mathrm{tr}\{(XW)(XW)'\}] \\
&= \mathsf{E}[\mathrm{tr}(XWW'X')] = \mathrm{tr}(\mathsf{E}[XWW'X]) \\
&= \mathrm{tr}(X\mathsf{E}[WW']X') = \mathrm{tr}(XQX')
\end{aligned}$$

subject to $XV = I$.