The Temporal Poisson Process

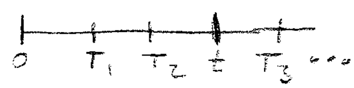
The Homogeneous Poisson Process

Let $X_1, X_2, \ldots$ be i.i.d. $\exp(\lambda)$ RVs. Define

$$T_k := \sum_{i=1}^{k} X_i$$

Then $T_k \sim \text{Erlang}(k, \lambda)$. Next put

$$N_t := \sum_{k=1}^{\infty} I_{[0,t]}(T_k) = \# T_k \le t.$$



Observe that since $T_1 < T_2 < \cdots$,

$$N_t = \max\{k \ge 1 : T_k \le t\}$$

Thus, $N_t = n \iff T_n \le t \;\&\; T_{n+1} > t$.

Next, since $N_t < n \iff T_n > t$. Hence,

$$P(N_t < n) = P(T_n > t) = \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

It follows that

$$P(N_t = n) = P(N_t < n+1) - P(N_t < n)$$
$$= \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

It can further be shown [Billingsley] that N_t has independent increments, i.e.,

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For $0 \le s_1 < t_1 \le s_2 < t_2 \le \cdots \le s_n < t_n < \infty$,

$N_{t_1} - N_{s_1}, \ldots, N_{t_n} - N_{s_n}$ are <u>independent</u>

with $\mathcal{P}(N_t - N_s = n) = \dfrac{[\lambda(t-s)]^n e^{-\lambda(t-s)}}{n!}$, $n = 0, 1, 2, \ldots$

Conversely, it can be shown [Billingsley] that if $N_0 \equiv 0$ and N_t has $\perp\!\!\!\perp$ Poisson increments, then $T_n := \min\{t \ge 0 : N_t \ge n\} \sim \text{Erlang}(n, \lambda)$, and $X_1 := T_1$, $X_k := T_k - T_{k-1}$ for $k \ge 2$ are $\perp\!\!\!\perp$ $\exp(\lambda)$ RVs.

In particular,

$$f_{T_n|T_{n-1},\ldots,T_1}(t|t_{n-1},\ldots,t_1) = f_{T_n|T_{n-1}}(t|t_{n-1})$$
$$= \lambda e^{-\lambda(t-t_{n-1})}, \quad t \ge t_{n-1}.$$

To see this, write

$$\mathcal{P}(T_n \le t | T_{n-1} = t_{n-1}, \ldots, T_1 = t_1)$$
$$= \mathcal{P}(T_{n-1} + X_n \le t | T_{n-1} = t_{n-1}, \ldots, T_1 = t_1)$$
$$= \mathcal{P}(t_{n-1} + X_n \le t | T_{n-1} = t_{n-1}, \ldots, T_1 = t_1)$$
$$= \mathcal{P}(X_n \le t - t_{n-1} | \underbrace{T_{n-1} = t_{n-1}, \ldots, T_1 = t_1}_{\text{depends on } X_1, \ldots, X_{n-1}})$$
$$= \mathcal{P}(X_n \le t - t_{n-1}), \text{ by } \perp\!\!\!\perp$$
$$= 1 - e^{-\lambda(t-t_{n-1})}.$$

Since $P(T_n \le t \mid T_{n-1} = t_{n-1}, \ldots, T_1 = t_1)$

depends only on t_{n-1} & not on $t_{n-2}, \ldots, t_1$, this conditional probability must be equal to $P(T_n \le t \mid T_{n-1} = t_{n-1})$. Now differentiate to get the result for the conditional densities.

The Inhomogeneous Poisson Process

We say that $\{N_t, t \ge 0\}$ is an inhomogeneous Poisson process if

(i) $N_0 = 0$

(ii) N_t has independent increments with

$$P(N_t - N_s = n) = \frac{\left[\int_s^t \lambda(\tau)\,d\tau\right]^n e^{-\int_s^t \lambda(\tau)\,d\tau}}{n!}$$

$n = 0, 1, 2, \ldots$

If J is an interval, say $J = [s,t]$, we put

$$\Lambda(J) := \int_s^t \lambda(\tau)\,d\tau$$

For example, if $J = [s,t]$,

$$P(N_t - N_s = n) = \frac{\Lambda(J)^n e^{-\Lambda(J)}}{n!}$$

We also put $N(J) := N_t - N_s$. Then

$$E[N(J)] = E[N_t - N_s] = \int_s^t \lambda(\tau)\,d\tau = \Lambda(J)$$

Given a Poisson process N_t with intensity or rate $\lambda(\tau)$, put

$$T_k := \min\{t \geq 0 : N_t \geq k\}.$$

Let us find the joint conditional density of $T_1, \ldots, T_n$ given $N_t = n$. We start with the joint probability,

$$P\left(T_1 \in [t_1, t_1+\Delta t_1), \ldots, T_n \in [t_n, t_n+\Delta t_n), N_t = n\right) \quad (*)$$

where $0 < t_1 < \cdots < t_n < t$ and

$$t_k + \Delta t_k < t_{k+1} \quad \text{for} \quad k = 1, \ldots, n-1$$

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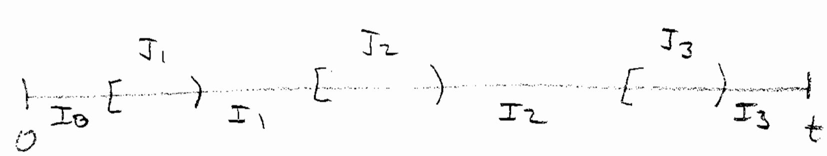
and $t_n + \Delta t < t$. Put $J_k := [t_k, t_k + \Delta t_k)$ and $I_0 := [0, t_1)$, $I_k := [t_k + \Delta t_k, t_{k+1})$ for $k = 1, \ldots, n$, where $t_{n+1} := t$. Then $(*)$ is equal to

$$P\left(\left[\bigcap_{k=1}^{n} \{N(I_k) = 0, N(J_k) = 1\}\right] \cap N(I_0) = 0\right) \quad (**)$$



By the $\perp\!\!\!\perp$ increments property, $(**)$ is equal to

$$\prod_{k=0}^{n} e^{-\Lambda(I_k)} \cdot \prod_{k=1}^{n} \Lambda(J_k)\, e^{-\Lambda(J_k)}$$

[GOTO p. TP-5]

Next, the desired conditional density is

$$\lim_{\Delta t_1 \downarrow 0, \ldots, \Delta t_n \downarrow 0} \frac{1}{\Delta t_1 \cdots \Delta t_n} \cdot \frac{P(T_1 \in [t_1, t_1+\Delta t_1), \ldots, T_n \in [t_n, t_n+\Delta t_n), N_t = n)}{P(N_t = n)} \qquad (***)$$

Of course,

$$P(N_t = n) = \frac{\Lambda([0,t])^n \, e^{-\Lambda([0,t])}}{n!}$$

To say more about the limit, we must analyze the joint probability. [Return to p. TP-5]

$$= \left[\prod_{k=1}^{n} \Lambda(J_k) \right] e^{-\left\{ \sum_{k=0}^{n} \Lambda(I_k) + \sum_{k=1}^{n} \Lambda(J_k) \right\}}$$

Now, $\dfrac{\Lambda(J_k)}{\Delta t_k} = \dfrac{1}{\Delta t_k} \displaystyle\int_{t_k}^{t_k+\Delta t_k} \lambda(\tau)\,d\tau \longrightarrow \lambda(t_k).$

Also,

$$\sum_{k=0}^{n} \Lambda(I_k) + \sum_{k=1}^{n} \Lambda(J_k) = \int_0^{t_1} \lambda(\tau)\,d\tau + \sum_{k=1}^{n} \int_{t_k}^{t_{k+1}} \lambda(\tau)\,d\tau$$

$$= \int_0^t \lambda(\tau)\,d\tau = \Lambda([0,t])$$

So, the limit in (***) is

$$n! \prod_{k=1}^{n} \frac{\lambda(t_k)}{\int_0^t \lambda(\tau)\,d\tau}, \quad 0 < t_1 < t_2 < \cdots < t_n < t$$

This is the conditional density of $T_1, \ldots, T_n$ given $N_t = n$.